The pool hits:

an analysis to the route of balls in pool

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Summary

Mathematics plays a pivotal role in all kinds of sports. The popular pool game is especially related to mathematics. This work studies the mathematics involved in the pool game. By consideration of physics and geometry, the routes of pool balls are calculated and the strategies to get target balls into pockets are evaluated.

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Abstract:

Our project is to find out the mathematics behind pool game, especially how it is related to the routes of object balls and the cue ball. For the object balls, the number of routes for a player to choose, and the direction of the object ball are determined. For the cue ball, different contact points on it result in different routes, due to the difference of spherical coordinates. Moreover, the routes of balls after collision are different from the original routes. The change of routes after collision is also studied. In this work, the concepts of mathematical induction, trigonometry, spherical coordinates and vectors are involved. Physic methods like reflection of force and moment of inertia are used to solve the problems in this project.

Results:

Part I

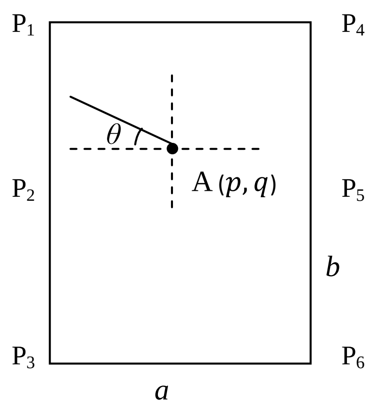

Fig.1: A pool table with six pockets

P_k, pocket *k;* Angle θ, the direction of the route of the object ball;

p, the distance between the object ball and the left vertical cushion;

q, the distance between the object ball and the lower horizontal cushion.

According to the reflection of force, the angle of reflection is equal to the angle of incidence. Thus, to find out the route of the object ball, the method is quite similar to that of finding the route of the light when it is reflected. We can construct an image of the pool table and then join the image of pocket 1 to the object ball. Hence, the route of the ball is found, namely AOP_1 (Fig. 2).

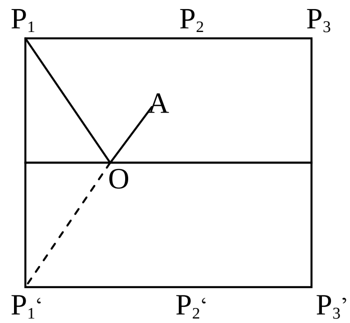

Fig.2: Reflection of the object

Let A hit the edges for *n* times before it goes into pocket *k* (*k*=1, 2, 3, 4, 5, 6) and assume the pattern can extend infinitely. It is obvious that this pattern is symmetrical about the lines AB, CD, EF and GH (which are the cushions of the table).

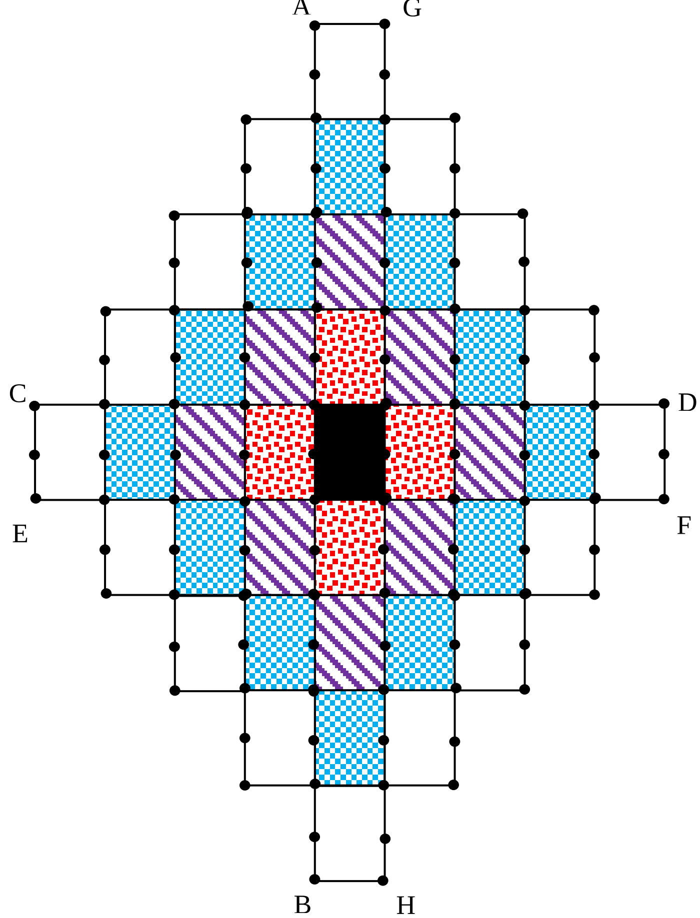

Fig.3: images of the pool table

The black rectangle in Fig.3, which represents the original pool table, is labelled as layer zero. The dotted rectangles around the black rectangle, which represent the first images of the pool table, are labelled as layer one. Layer one is defined excluding the four corners of layer zero. Similarly, layer *n* consists of rectangles, which represent the n^{th} images of the pool table, around layer (*n*-1). It does not include the points which are already in layer (*n*-1).

If a ball in the black rectangle can go into one of the images of pocket *k* on layer *n* in a straight line, then it can go into the corresponding image of pocket *k* on layer (*n*-1) after being reflected by one of the edges of the rectangle for once. This is because the rectangles on the figure are all symmetrical about their edges.

By mathematical induction, if an object ball in the original pool table can get into the image of pocket *k* on layer *n* in a straight line, then it can go into pocket *k* on layer zero after being reflected by line AB, CD, EF or GH for *n* times.

According to this result, the number of routes for pockets 1, 3, 4 and 6 is (*n*+1) and the number of routes for pockets 2 and 5 is (2*n*+1). (See detailed procedure in **Appendix 2**)

Part II

Let the length of the horizontal cushion be *a*, the length of the vertical cushion be *b*, the distance between point A and the lower horizontal cushion be *q*, and the distance between A and the left vertical cushion be *p* (Fig.1).

Since the four corner pockets 1, 3, 4 and 6 are identical, the content below will only study the direction of the object ball to pocket 1. Similarly, for middle pockets, only pocket 2 will be studied.

If the ball rebounds at the horizontal cushions for *i* times and at the vertical cushions for *j* times (*i* and *j* are non-negative integers), we find that (See detailed procedure in **Appendix** 3):

For the corner pockets:

When *i* and *j* are both even,

$$\theta = tan^{-1}\frac{b(i+1)-q}{aj+p}\ (0° < \theta < 90°)$$

When *i* is even and *j* is odd,

$$\theta = tan^{-1}\frac{-b(i+1)+q}{a(j+1)-p}\ (90° < \theta < 180°)$$

When *i* and *j* are both odd,

$$\theta = tan^{-1}\frac{bi+q}{a(j+1)-p}\ (180° < \theta < 270°)$$

When *i* is odd and *j* is even,

$$\theta = tan^{-1}\frac{-b(i+1)-q}{a(j-1)+p}\ (270° < \theta < 360°)$$

For the middle pockets:

Just replace every q in the above equations with $(q+\frac{1}{2})$.

Part III

This part will specifically study the route of the cue ball.

In Figure.4, the centre of the ball is the origin. The *x*-axis and *y*-axis are horizontal while the *z*-axis is vertical. The *x*-axis is in the same vertical plane as the initial force, *F*, imposed on the ball. Let the cue hit the ball at the point (r, φ, γ), which is in the right half of the ball with an angle of α to the horizon.

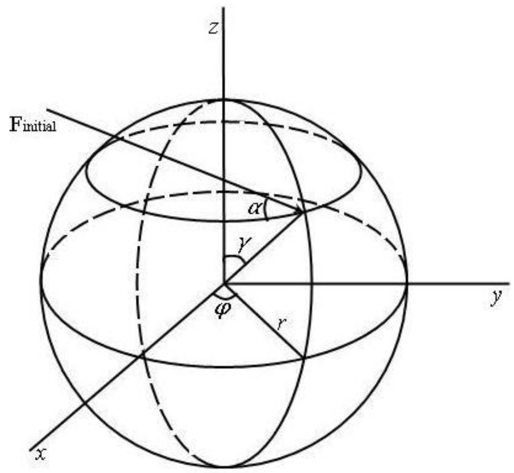

Fig.4: the initial force on the cue ball

By consideration of frictional force, angular momentum and composition of forces, the following results are obtained (See detailed procedure in **Appendix 4**):

1) When the contact point is on the lower half of the ball, the displacement of the ball along the direction of y-axis according to time is $d_y = v_{initial}\, t + (\, M_m - f_h^{'})t^2/2m$. Convert all the forces related to the original force applied on the cue ball back, we have:

$d_y = v_{initial}\, t + [K(W + cos\,\gamma\, F) - cos\,\beta\, \mu_0^{'}(W + cos\,\gamma\, F)]t^2/2m$;

Similarly, the displacement of the ball along the direction of x-axis according to time is $d_x = f_p^{'}\, t^2/2m$, which is actually: $d_x = sin\,\beta\, \mu_0^{'}(W + cos\,\gamma\, F)t^2/2m$.

2) When the contact point is on the upper half of the ball, the displacement of the ball along the direction of y-axis according to time is $d_y = v_{initial}\, t + (\, f_h^{'} - M_m)t^2/2m$, which is: $d_y = v_{initial}\, t + [cos\,\beta\, \mu_0^{'}(W + cos\,\gamma\, F) - K(W + cos\,\gamma\, F)]t^2/2m$;

Similarly, the displacement of the ball along the direction of x-axis according to time is $d_x = f_p^{'}\, t^2/2m$, which is actually $d_x = sin\,\beta\, \mu_0^{'}(W + cos\,\gamma\, F)t^2/2m$.

Part IV

In this part, the motions of balls after collision will be studied. If a collision between two balls or between a ball and the cushion could exist (See details about detecting whether a collision exists in **Appendix 5**), then we can use conservation of energy, linear momentum and angular momentum to calculate the route of balls after collision, assuming the collisions are perfectly elastic.

The relative linear velocity at the point of contact of ball 1 with respect to ball 2 is found to be (see detailed procedure in **Appendix 6**):

$$\overrightarrow{v_{LR}} = r \times (\overrightarrow{\omega_1} - \overrightarrow{\omega_2})$$

Then the frictional force between the balls at the moment of collision is

$$\overrightarrow{F_f} = -\frac{\mu m \Delta v_{n1}}{\Delta t} \cdot \frac{\overrightarrow{v_{LR}} + \overrightarrow{v_{t1}}}{|\overrightarrow{v_{LR}} + \overrightarrow{v_{t1}}|}, \Delta v_{n1} = |\overrightarrow{v_{n1}}| + |\overrightarrow{v_{n2}}|.$$

Since $\overrightarrow{\tau_1} = I\overrightarrow{\alpha_1}$,

$$\overrightarrow{r_1} \times \overrightarrow{F_f} = \frac{2}{5} m r^2 \cdot \frac{\Delta \omega_1}{\Delta t}$$

$$\Delta \omega_1 = \frac{5}{2} \overrightarrow{F_f} \frac{\Delta t}{m r^2} \overrightarrow{r_1}.$$

Similarly, for ball 2,

$$\Delta \omega_2 = \frac{5}{2} \overrightarrow{F_f} \frac{\Delta t}{m r^2} \overrightarrow{r_2}.$$

After finding out the change in angular velocity, the final angular velocity after collision can be found out.

By using the result of Part III, we then can find out the route of the balls after collision.

Since there is no external force applied, $F = 0$; and β can be found out through integrate horizontal component of $\overrightarrow{\omega_{final}}$ with respect to time.

Thus , displacement along the direction of the normal of contact is

$$d_y = |\overrightarrow{v_1}|t + (\mu_0^{'} cos\,\beta - k)\frac{t^2\omega}{2m};$$

while displacement along the direction of the normal of contact is

$$d_x = \frac{t^2}{2m}\mu_0^{'}\omega sin\,\beta.$$

Conclusion and Discussion:

From the above results, we find that the more times the object ball rebounds, the greater number of choices of the ball's direction a player has. However, when the number of routes increases; the differences between two consecutive angles decrease. Moreover, the general conditions of movement of the cue ball are due to the difference of contact points. However, the route of the cue ball will be affected by many factors other than the contact point, i.e. the material of the cue ball, the cue itself, the pool table and etc. Furthermore, the routes of balls after collision can also be calculated if the original routes and speeds of the balls are known.

Our findings can help players to choose a suitable route and a winning strategy. More than that, the method of constructing images can also be applied in other areas involving reflections. This will make calculations much easier.

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Recommendation for further studies:

The following questions remained unsolved due to the limitation of our knowledge and time:

1. What will the routes be if 3 or more balls collide at the same instant?
2. How to find the contact point between the cue ball and the object ball to pocket the object ball in?
3. Can a jump ball be made easily? What are the factors affecting the jumped ball? How these factors are mathematically related?

The answers to these questions would enhance our understanding of the route of balls in pool game.

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Appendix 1---Terminology:

Pocket: a pouch or net at each corner and side of a pool table

Cue: a long tapering stick used to strike the cue ball

Cue ball: the white ball struck with the cue so that it strikes the object ball in turn

Object ball: the ball that a player intends to hit with the cue ball

Cushion: the raised rim around the top of a pool table that boards its playing surface

Contact point: the point that the cue hits the cue ball

Jump ball: the cue ball that jumps over an object ball before hitting another object ball

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Appendix 2---Procedure of Part I:

If the object ball hits the cushion for *n* times, then the route of the ball is the joint line of the n^{th} image of the pocket and the ball. Fig. 4 shows the pattern:

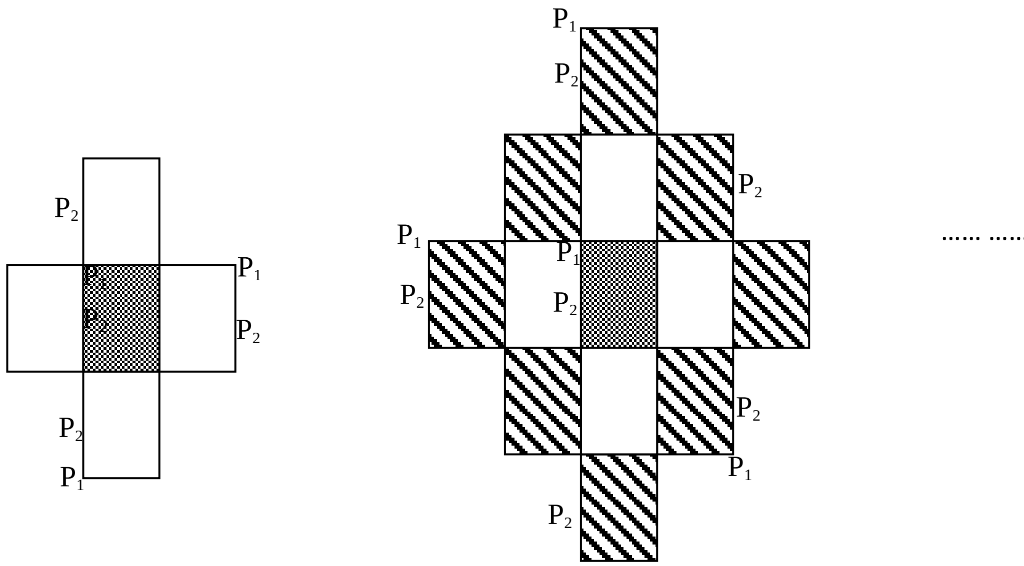

Fig.5: no. of routes for pockets 1&2 when *n*=1, 2......

From Fig. 4, we can find patterns for pocket 1 and pocket 2 respectively:

n^{th} image	No. of n^{th} images	No. of routes	
		Pocket 1	Pocket 2
1	2	2	3
2	3	3	5
3	4	4	7
4	5	5	9
5	6	6	11
…	…	…	…

Table 1: no. of routes against no. of images

From the results shown on table 1, we induced that for the n^{th} image, there are (n+1) images for each pocket.

Appendix 3---Procedure of Part II:

For the corner pockets:

If the ball hits the horizontal cushion for an even number of times and then gets into pocket 1, the horizontal image of pocket 1 is above the original table. If the ball hits the horizontal cushion for an odd number of times, the horizontal image of pocket 1 is below the original table. Similarly, if the ball hits the vertical cushion for an even number of times, the vertical image of pocket 1 is at the left of the original table. If the ball hits the vertical cushion for an odd number of times, then the vertical image of pocket 1 is at the right.

If $n = 2k, (k \text{ is non} - \text{negative integer})$, we have formulae below:

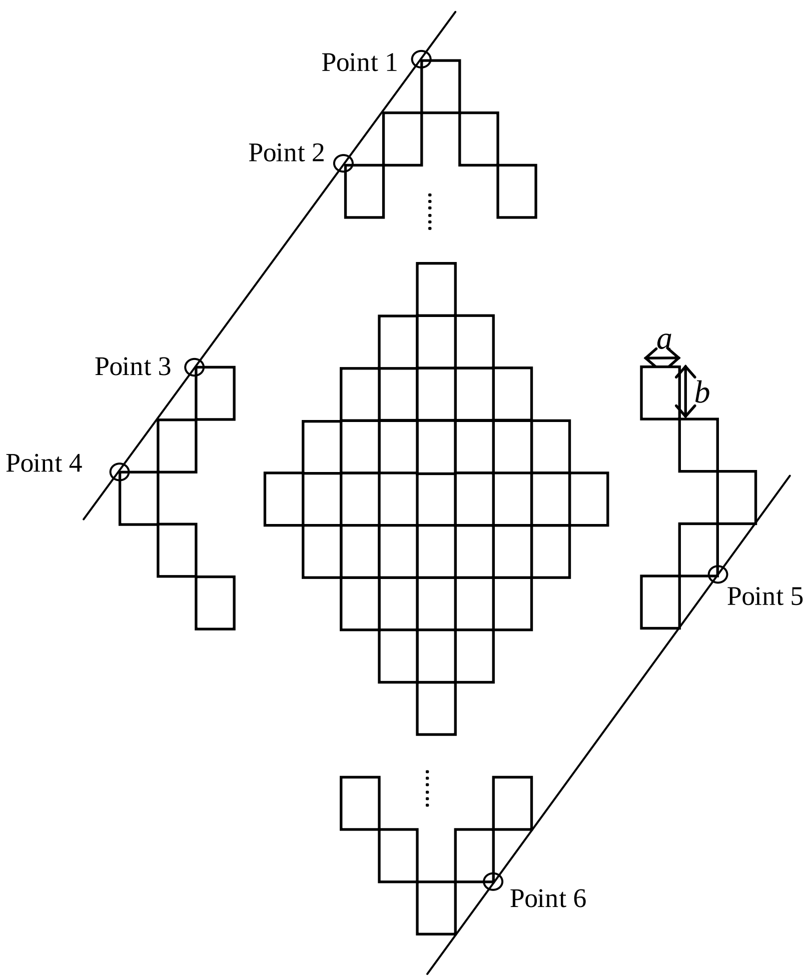

Fig.6: images of pocket 1 when *n* is

When the images are in the first quadrant, all θ here are $0° < \theta < 90°$:

For point 4,

$$\theta = tan^{-1}\frac{b-q}{an+p}$$

For point 3,

$$\theta = tan^{-1}\frac{3b-q}{a(n-2)+p}$$

……

For point 2,

$$\theta = tan^{-1}\frac{b(n-1)-q}{2a+p}$$

For point 1,

$$\theta = tan^{-1}\frac{b(n+1)-q}{p}$$

&

When the images are in the third quadrant, all θ here are $180° < \theta < 270°$:

For point 5,

$$\theta = tan^{-1}\frac{b+q}{a(n-1)+p}$$

Then,

$$\theta = tan^{-1}\frac{3b+q}{a(n-3)+p} ,$$

$$\theta = tan^{-1}\frac{5b+q}{a(n-5)+p} ,$$

……

For point 6,

$$\theta = tan^{-1}\frac{bn+q}{p}$$

If $n = 2k + 1$, (k is non − negative integer) , the following results are formulated:

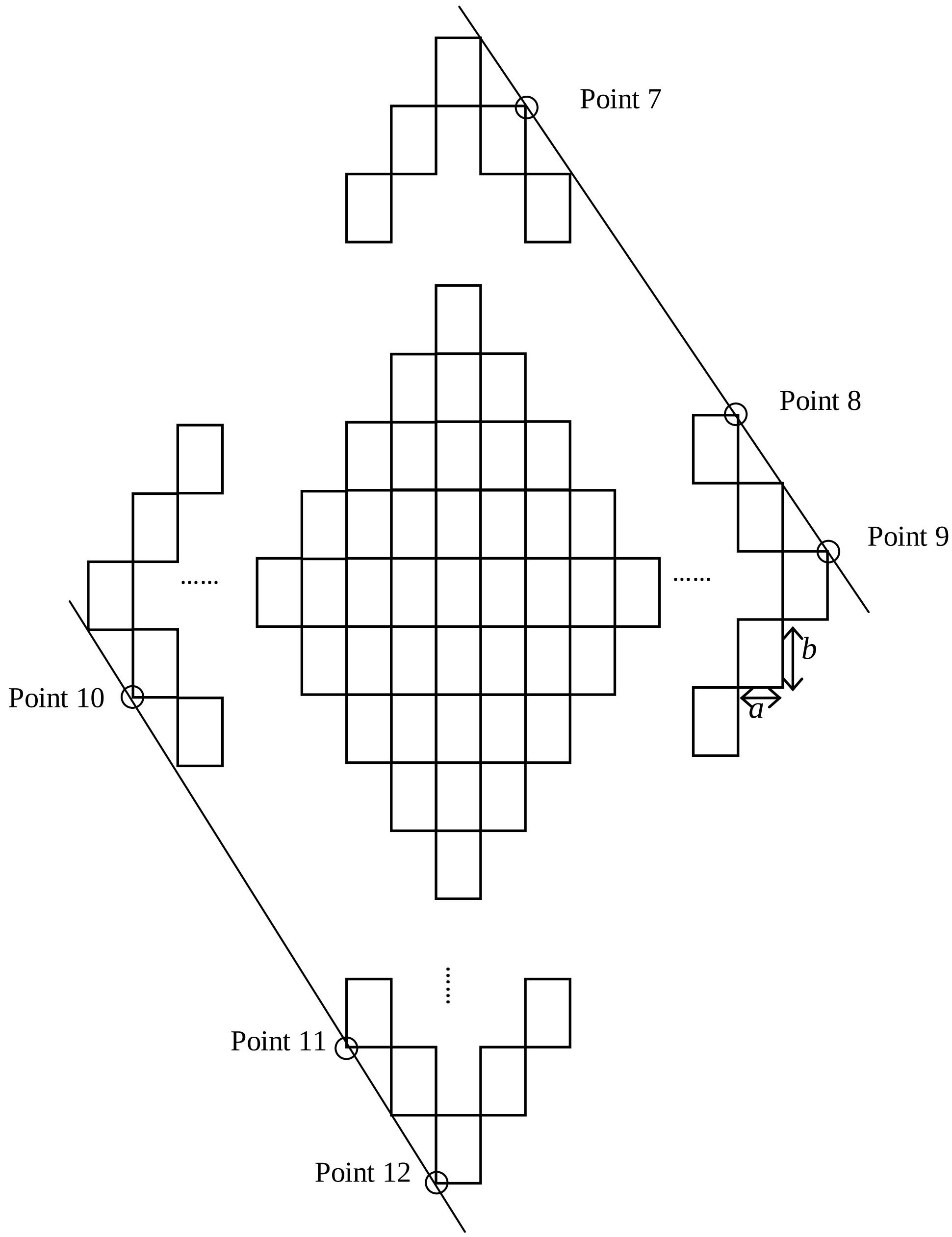

Fig.7: images of pocket 1 when n is odd

When the images are in the second quadrant, all θ here are $90° < \theta < 180°$:

For point 9,

$$\theta = tan^{-1}\frac{q-b}{a(n+1)-p}$$

For point 8,

$$\theta = tan^{-1}\frac{q-3b}{a(n-1)-p}$$

Then,

$$\theta = tan^{-1}\frac{q-5b}{a(n-3)-p}$$

……

For point 7,

$$\theta = tan^{-1}\frac{q-bn}{2a-p}$$

&

When the images are in the fourth quadrant, all θ here are $270° < \theta < 360°$:

For point 10,

$$\theta = tan^{-1}\frac{-b-q}{a(n-1)+p}$$

For point 11,

$$\theta = tan^{-1}\frac{-3b-q}{a(n-3)+p}$$

Then,

$$\theta = tan^{-1}\frac{-5b-q}{a(n-5)+p}$$

……

For point 12,

$$\theta = tan^{-1}\frac{-bn-q}{p}$$

For the middle pockets:

When the images of pocket 2are above the original pool table:

$$\theta = tan^{-1}\frac{b\left(n+\frac{1}{2}\right)-q}{p}$$

$$\theta = tan^{-1}\frac{b\left(n-\frac{1}{2}\right)-q}{2a-p}$$

$$\theta = tan^{-1}\frac{b\left(n-\frac{3}{2}\right)-q}{2a+p}$$

…

$$\theta = tan^{-1}\frac{\frac{b}{2}-q}{an\pm p}$$

(When *n* is odd, take the symbol “—”; when *n* is even, take the symbol “+”.)

&

When the images of pocket 2 are below the original pool table:

$$\theta = tan^{-1}\frac{b\left(n-\frac{1}{2}\right)+q}{p}$$

$$\theta = tan^{-1}\frac{b\left(n-\frac{3}{2}\right)+q}{2a-p}$$

$$\theta = tan^{-1}\frac{b\left(n-\frac{5}{2}\right)+q}{2a+p}$$

…

$$\theta = tan^{-1}\frac{\frac{b}{2}+q}{an\pm p}$$

(When *n* is odd, take the symbol “—”; when *n* is even, take the symbol “+”.)

Appendix 4---Procedure of Part III:

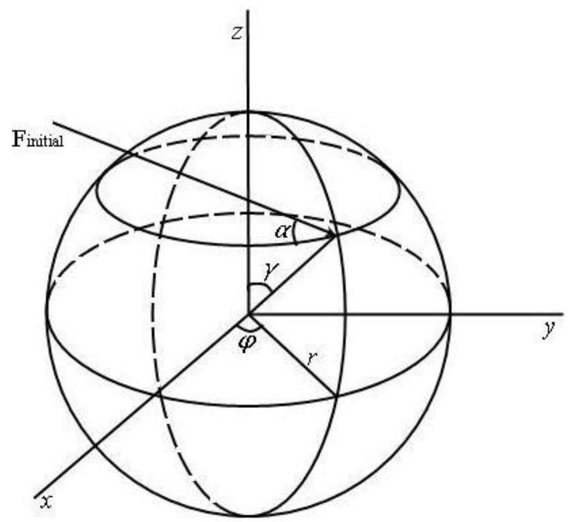

Fig.4: the initial force on the cue ball

The force exerting on the ball can be degraded into one horizontal component and one vertical component. The horizontal component has the same direction as x-axis, which is $F_H = sin\,\gamma\, F$; the vertical component is $F_V = cos\,\gamma\, F$.

Also, when the cue is in contact with the ball, a sliding frictional force between the cue and the ball is produced, which is $f = F\mu_0$ (μ_0 is the sliding frictional coefficient). The direction of the frictional force is in y-axis and with an angle of α to the horizontal. Thus, the frictional force can be degraded into two components as well. The horizontal component is $f_H = cos\,\alpha\, f = cos\,\alpha\, F\mu_0$ and the vertical one is $f_V = sin\,\alpha\, f = sin\,\alpha F\mu_0$.

Thus, as moving along the direction of x–axis, the ball spins vertically and horizontally. In total, the ball spins in a direction which has an angle of β to the x-axis. However, the ball still moves in the direction of x–axis.

More importantly, angle β is decided by the horizontal angular speed only.

Since τ, the torque of the ball is

$$\tau = r\, f_H = I\, \beta / t^2, (t \text{ is the time duration of contacting})$$

And I, moment of inertia of the solid ball (R Nave, 2005), according to the equation, is

$$I = \frac{2}{5} m r^2$$

β is found to be:

$$\beta = \frac{5\, f_H t^2}{2mr}$$

So there will be a sliding frictional force between the table and the ball, when it is spinning. The sliding frictional force is $f' = \mu_0'(W + F_V)$, and its two components in *x*-axis and *y*-axis are f_h' & f_p' respectively. $f_h' = cos\, \beta\, f', f_p' = sin\, \beta\, f'$.

Also, when the ball rotates, there will be a rolling frictional force opposing the direction of rotating. The rolling frictional force is
$M_m = K(W + F_V), (K \text{ is the rolling frictional coefficient})$.

Appendix 5---Procedure of detecting the existence of collision in Part IV

First of all, we need to detect whether the collision between 2 balls will exist. To detect the collision, the equation of a line is used to describe the positions of the balls (assuming the frictional force is negligible):

$$\vec{\boldsymbol{x}} = \overrightarrow{\boldsymbol{x_0}} + \vec{\boldsymbol{v}}t$$

$\vec{\boldsymbol{x}}$: the position of the ball at time t

$\overrightarrow{\boldsymbol{x_0}}$: the initial position (in subsequent part $\overrightarrow{x_{01}}$ represents the initial position of ball 1, while $\overrightarrow{x_{02}}$ the one of ball 2)

$\vec{\boldsymbol{v}}$: velocity of the ball (in subsequent part $\overrightarrow{v_1}$ represents the velocity of ball 1, while $\overrightarrow{v_2}$ the one of ball 2)

t: time period of the ball after initial position and before any collision.

Thus, at just one moment before collision, between 2 balls (one ball can be at rest initially),

$$\sqrt{(\overrightarrow{x_1} - \overrightarrow{x_2}) \cdot (\overrightarrow{x_1} - \overrightarrow{x_2})} = 2r$$

$$\sqrt{(\overrightarrow{x_{01}} - \overrightarrow{x_{02}} + \overrightarrow{v_1}t - \overrightarrow{v_2}t) \cdot (\overrightarrow{x_{01}} - \overrightarrow{x_{02}} + \overrightarrow{v_1}t - \overrightarrow{v_2}t)} = 2r$$

$$\sqrt{(\overrightarrow{x_{01}} - \overrightarrow{x_{02}}) \cdot (\overrightarrow{x_{01}} - \overrightarrow{x_{02}}) + 2(\overrightarrow{x_{01}} - \overrightarrow{x_{02}}) \cdot (\overrightarrow{v_1} - \overrightarrow{v_2})t + (\overrightarrow{v_1} - \overrightarrow{v_2}) \cdot (\overrightarrow{v_1} - \overrightarrow{v_2})t^2} = 2r$$

To simplify, let

$$a = (\overrightarrow{v_1} - \overrightarrow{v_2}) \cdot (\overrightarrow{v_1} - \overrightarrow{v_2})$$

$$b = 2(\overrightarrow{x_{01}} - \overrightarrow{x_{02}}) \cdot (\overrightarrow{v_1} - \overrightarrow{v_2})$$

$$c = (\overrightarrow{x_{01}} - \overrightarrow{x_{02}}) \cdot (\overrightarrow{x_{01}} - \overrightarrow{x_{02}})$$

$$\therefore at^2 + bt + c = 4r^2$$

Thus if t is a real number,

$$b^2 - 4a(c - 4r^2) \geq 0$$

If the above inequality is satisfied, then collision between 2 balls can occur. If not, the collision cannot occur.

If frictional force is considered, only $\vec{v}$ will be different.

$\vec{v}$ then will be $\left(\frac{d}{dt}d_x, \frac{d}{dt}d_y\right)$, ($d_x, d_y$ are results calculated in Part III). More specifically, it is $(\frac{f'_p t}{m}, v_{initial} + \frac{(f'_h - M_m)t}{m})$.

Appendix 6---Procedure of calculating relative linear velocity in Part IV

Firstly, let $\vec{n} = \frac{\vec{x_1}-\vec{x_2}}{|\vec{x_1}-\vec{x_2}|}$. $\vec{n}$ is the unit vector according to the positions of the balls at the moment of collision.

If the velocity of the 2 balls at the moment of collision are $\vec{v_1}$ and $\vec{v_2}$ respectively, then the normal components of the velocities are

$$\overrightarrow{v_{n1}} = \left(\vec{v_1} \cdot (-\vec{n})\right)(-\vec{n})$$

$$\overrightarrow{v_{n1}} = (\vec{v_1} \cdot \vec{n})\vec{n}$$

Thus, the tangential components are

$$\overrightarrow{v_{t1}} = \overrightarrow{v_{n1}} - \vec{v_1}$$

$$\overrightarrow{v_{t2}} = \overrightarrow{v_{n2}} - \vec{v_2}$$

According to conservation of energy and conservation of linear momentum:

$$m\overrightarrow{v_{n1}} + m\overrightarrow{v_{n2}} = m\overrightarrow{v'_{n1}} + m\overrightarrow{v'_{n2}}$$

$$\frac{1}{2}m|\overrightarrow{v_{n1}}|^2 + \frac{1}{2}m|\overrightarrow{v_{n2}}|^2 = \frac{1}{2}m|\overrightarrow{v'_{n1}}|^2 + \frac{1}{2}m|\overrightarrow{v'_{n2}}|^2$$

$$\Rightarrow \left(\overrightarrow{v'_{n1}} - \overrightarrow{v_{n1}}\right) \cdot \left(\overrightarrow{v'_{n1}} - \overrightarrow{v_{n2}}\right) = 0$$

Since $\overrightarrow{v_{n1}} \neq \overrightarrow{v'_{n1}}$ *and* $\overrightarrow{v_{n2}} \neq \overrightarrow{v'_{n2}}$,

$$\overrightarrow{v'_{n1}} = \overrightarrow{v_{n2}}, \overrightarrow{v'_{n2}} = \overrightarrow{v_{n1}}$$

$$\Rightarrow \overrightarrow{v'_1} = \overrightarrow{v_{n2}} + \overrightarrow{v_{t1}}, \overrightarrow{v'_2} = \overrightarrow{v_{n1}} + \overrightarrow{v_{t1}},$$

To calculate the angular velocity, it is changed by the frictional force at the moment of collision when the balls come into contact.

Thus, the relative linear velocity at the point of contact of ball 1 with respect to ball 2 is:

$$\overrightarrow{v_{LR}} = r \times (\vec{\omega_1} - \vec{\omega_2})$$