Roman Domination

WANG Shizhi
Hwa Chong Institution

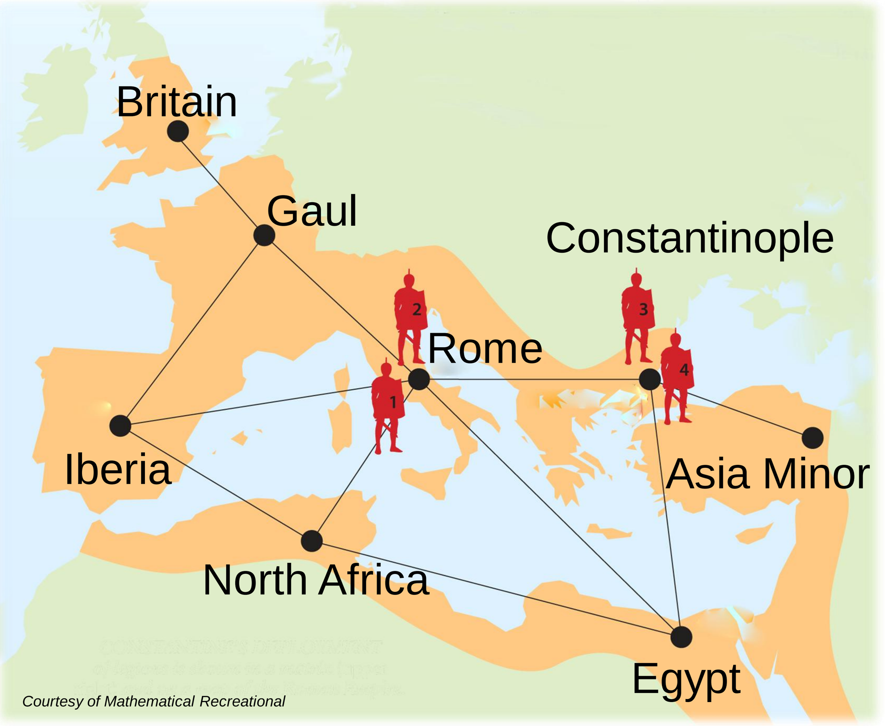
Mentor: Prof. KOH Khee Meng
National University of Singapore

History and Motivation

How can Emperor Constantine station his four field army units to protect eight regions?

His trick was to place the army units so that every region is,

- secured by its own army, or
- securable by a neighbor with two army units (one of which can be sent to the undefended region directly if a conflict breaks out).



The same trick can also be used for optimizing the location of

- declining number of British Fleets at the end of 19th century
- American Military Units during the Cold War (2).

Definitions

Let $G = (V, E)$ be a graph.

- **Roman dominating function**: a function $f : V \to \{0, 1, 2\}$ such that every vertex v for which $f(v) = 0$ has a neighbor u with $f(u) = 2$.
- **Weight** of a Roman dominating function f, $w(f) = \sum_{v \in V} f(v)$, corresponding to the total number of army units required under a specific deployment scheme – a function f.
- $\boldsymbol{\gamma_R}$**-function**: the Roman dominating function(s) of minimum weight among all the possible Roman dominating functions.
- The **Roman domination number** of a graph G, denoted by $\gamma_R(G)$, is the weight of γ_R-function(s) – the minimum weight of all possible Roman dominating functions.

Roman Dominating Index

- Theoretically we are concerned with adding or deleting an edge and how it will affect the Roman domination number of a graph.
- In practice, armies, utility operators, etc are concerned about where to build a new road, a pipeline, etc so as to reduce the size of army or reap the most economic benefits.

Thus we have a new concept called Roman dominating index, which is useful in simplifying some Roman domination problems.

Let G be a graph and x, y two non-adjacent vertices in G.

The **Roman dominating index** of $\{x, y\}$, denoted by $R(x, y)$, is defined by $R(xy) = \gamma_R(G) - \gamma_R(G + xy)$.

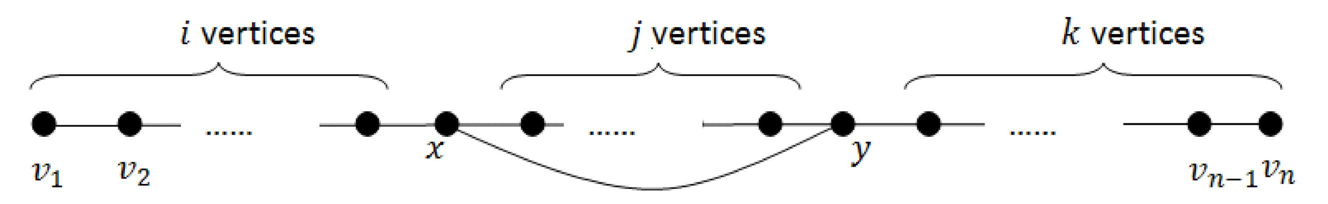
Proposition: Let G be a graph. For any pair of non-adjacent vertices $\{x, y\}$ in G, $0 \leq R(xy) \leq 1$.

Remark: Both the lower and upper bounds are reachable.

Corollary: Let $\{x, y\}$ be a pair of non-adjacent vertices in a graph G. Then $R(xy) = 1$ if and only if there exists a γ_R-function f of G such that $\{f(x), f(y)\} = \{1,2\}$.

Applications of Roman Dominating Index

Problem 1: Given a path P_n of order $n \geq 3$, are there pairs of non-adjacent vertices in P_n such that $R(xy) = 1$? If yes, which pairs?



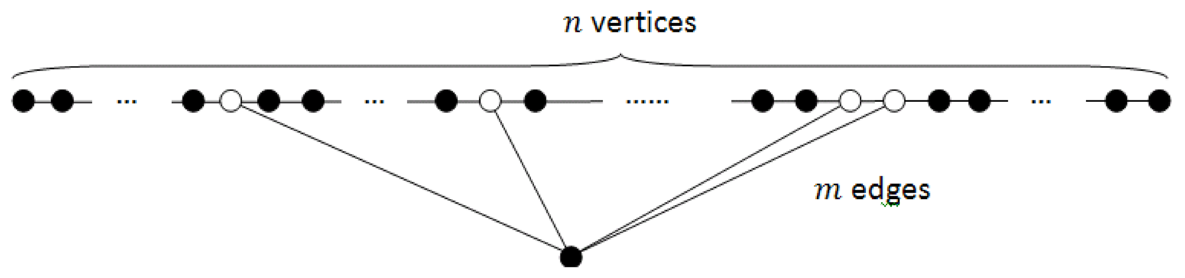
Result:

cases	exist $(xy) = 1$?	which pairs[1]?
$n \equiv 0 (\text{mod } 3)$	no	N.A.
$n \equiv 1 (\text{mod } 3)$,	yes	$i \equiv 0, j \equiv 1, k \equiv 1\ (\text{mod } 3)$
$n \equiv 2 (\text{mod } 3)$,	yes	$i \equiv 0, j \equiv 1, k \equiv 2$ or $i \equiv 1, j \equiv 1, k \equiv 1$ or $i \equiv 0, j \equiv 2, k \equiv 1\ (\text{mod } 3)$.

[1]Let f_{γ_R} be a γ_R-function for P_n. Assuming WLOG that $f_{\gamma_R}(x) = 1$ and $f_{\gamma_R}(y) = 2$.

Remark: By similar argument, we can show whether and how $R(xy) = 1$ can be achieved for some other classes of graphs.

Problem 2: Given a path P_n of order $n \geq 3$, a positive integer m with $m \leq n$, and a vertex v not in P_n, how to add m new edges to join v and m vertices in P_n so that the resulting graph G has the largest $\gamma_R(G)$? What is the value of this largest $\gamma_R(G)$? What about the smallest one?



Result:

cases	sub-cases	$\gamma_R(G)$	$f(v)$
largest	$m \leq \left\lfloor \frac{n+1}{3} \right\rfloor + 1$	$\left\lceil \frac{2n+2}{3} \right\rceil$	$\begin{cases} 1, \text{if } n \equiv 0\ or\ 1\ (mod3) \\ 0, \text{if } n \equiv 2\ (mod3) \end{cases}$
	$m \geq \left\lfloor \frac{n+1}{3} \right\rfloor + 1$	$n - m + 2$	2
smallest	$m \leq 3$	$\left\lceil \frac{2n}{3} \right\rceil$	0
	$m \geq 3$	$\left\lceil \frac{2}{3}(n-m) \right\rceil + 2$	2

Bound of Roman Domination Number as a Function of Order

Proposition: For any tree T of order $n \geq 3$, $2 \leq \gamma_R(T) \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

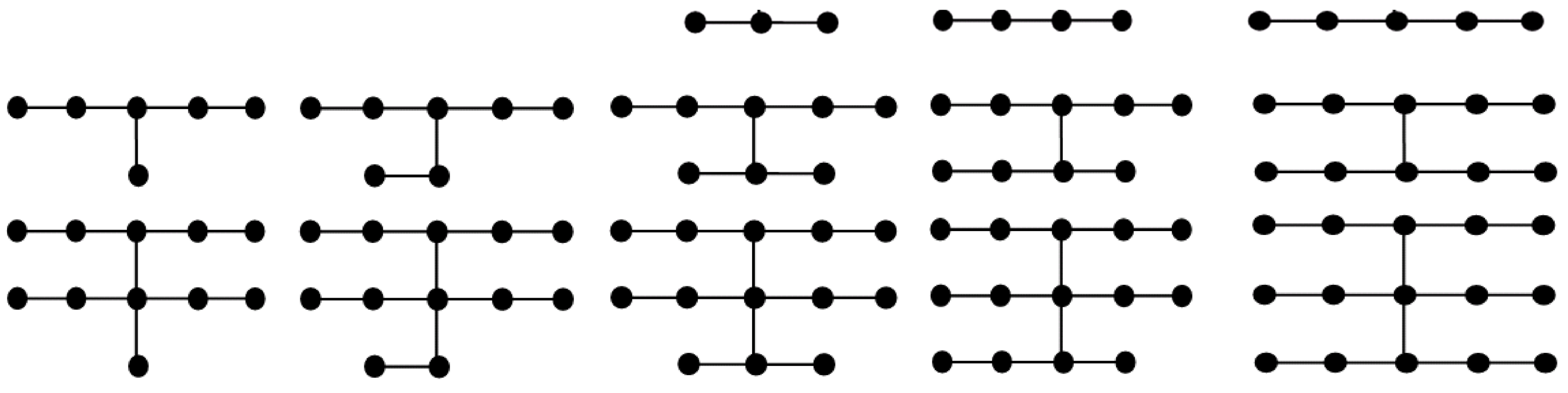
Proof sketch: The lower bound is trivial as no matter how large the order is, a star always has a Roman domination number of 2.

The upper bound is proven by **mathematical induction** on the diameter of tree, $D(T)$.

Base case: if $D(T) = 2, 3,$ or 4, $\gamma_R(T) \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

Inductive hypothesis: If $\gamma_R(T) \leq \left\lfloor \frac{4n}{5} \right\rfloor$ for any tree T of $k - 3 \leq D(T) \leq k - 1$, then for any tree T of $D(T) = k$, $\gamma_R(T) \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

Remark: This bound is achievable by constructing trees of the following structures.



Corollary: For any connected graph G of order $n \geq 3$, $2 \leq \gamma_R(G) \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

Photo Credit

Unless otherwise stated, all graphics on this poster are original.

Acknowledgement

Prof. Koh, thank you so much for your guidance and patience! I will remember your favorite line: do the right thing at the right time.

I would also like to thank Science Research Program, Hwa Chong Institution and Stanford University Math Camp for offering me so many opportunities.

Bibliography

1. **Stewart, Ian.** Defend the Roman Empire. *Scientific American.* December 1999, pp. 136-138.
2. **ReVelle, Charles S. and Rosing, Kenneth E.** Defendens Imperium Romanum: A Classical Problem in Military Strategy. *The American Mathematical Monthly, Vol. 107, No. 7.* Aug. - Sep. 2000, pp. 585-594.
3. **Dreyer, Paul Andrew.** *Center for Discrete Mathematics & Theoretical Computer Science.* [Online] October 2000. [Cited: December 15, 2008.] http://dimacs.rutgers.edu/~pdreyer/thesis.pdf.
4. **Koh, Khee Meng, Dong, Fengming and Tay, Eng Guan.** *Introduction to Graph Theory: H3 Mathematics.* Singapore : World Scientific Publishing Co. Pte. Ltd., 2007.