Roman Domination

Wang Shizhi

Wednesday, December 31, 2008

Contents

Acknowledgement

Special thanks to my project mentor, Prof. Koh Khee Meng, for his guidance. It is truly a privilege and an enriching journey to be mentored by him. I learned much more than what I have learned in my classroom. Thank you, Prof.Koh!

Structure of the paper

Chapters 1 to 4, 8 pages in total, are the core of this paper.

Chapter 1 and Chapter 2 present some backgrounds.

Chapter 3 and Chapter 4 are original.

Summary

During World War II, when General Douglas MacArthur was conducting military operations in the Pacific theater, he adopted a strategy of "island-hopping"— moving troops from one island to a nearby one, but only when he could leave behind a large enough garrison to keep the first island secure. A similar deployment problem faced the Roman emperor Constantine in the fourth century A.D., only his task was to maintain the security of an entire empire. He decided on what appears to be the first recorded use of the strategy that MacArthur later adopted in the Pacific (1). This paper models this strategy mathematically and presents some new results.

Abstract

In his article "Defend the Roman Empire!" (1999), Ian Stewart discussed a strategy of Emperor Constantine for defending the Roman Empire. Motivated by this article, Cockayne et al. (2004) introduced the notion of Roman domination in graphs.

Let $G = (V, E)$ be a graph. A Roman dominating function of G is a function $f : V \to \{0, 1, 2\}$ such that every vertex v for which $f(v) = 0$ has a neighbor u with $f(u) = 2$. The weight of a Roman dominating function f is $w(f) = \sum_{v \in V} f(v)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of all possible Roman dominating functions.

This paper introduces a quantity $R(xy)$ for each pair of non-adjacent vertices $\{x, y\}$ in G, called the Roman dominating index of $\{x, y\}$, which is defined by $R(xy) = \gamma_R(G) - \gamma_R(G + xy)$. We prove that $0 \leq R(xy) \leq 1$ and give a necessary and sufficient condition on $\{x, y\}$ for which $R(xy) = 1$.

This result is applied to find

1. given a path P_n of order $n \geq 3$ or a cycle C_n, which pairs of non-adjacent vertices $\{x, y\}$ are such that $R(xy) = 1$, and
2. given a path P_n of order $n \geq 3$, a positive integer m with $m \leq n$, and a vertex v not in P_n, the way to add m new edges to join v and m vertices in P_n so that the resulting graph G has the largest/smallest $\gamma_R(G)$. The largest $\gamma_R(G)$ is found to be $\begin{cases} \left\lceil \frac{2n+2}{3} \right\rceil, \left(m \leq \left\lfloor \frac{n+1}{3} \right\rfloor + 1\right) \\ n - m + 2, \left(m \geq \left\lfloor \frac{n+1}{3} \right\rfloor + 1\right) \end{cases}$, while the smallest one $\begin{cases} \left\lceil \frac{2n}{3} \right\rceil, (m \leq 3) \\ \left\lceil \frac{2}{3}(n-m) \right\rceil + 2, (m \geq 3) \end{cases}$.

Finally, we show that for each connected graph G of order $n \geq 3$, $2 \leq \gamma_R(G) \leq \left\lfloor \frac{4n}{5} \right\rfloor$. A family of graphs for which the respective equality holds is also provided.

1. History and Motivation

About 1700 years ago, the Roman Empire was under attack, and Emperor Constantine had to decide where to station his four field army units to protect eight regions. His trick was to place the army units so that every region is either secured by its own army (one or two units) or is securable by a neighbor with two army units, one of which can be sent to the undefended region directly if a conflict breaks out.

Constantine chose to place two army units in Rome and two at his new capital, Constantinople. This meant only Britain could not be reached in one step. As it happens, Constantine's successors lost control of Britain. The causes were surely more complex than anything that can be explained by this simple model. Nevertheless, Stewart (1) is right in arguing that if Constantine had been a better mathematician, the Roman Empire might have lasted a little longer than it did.

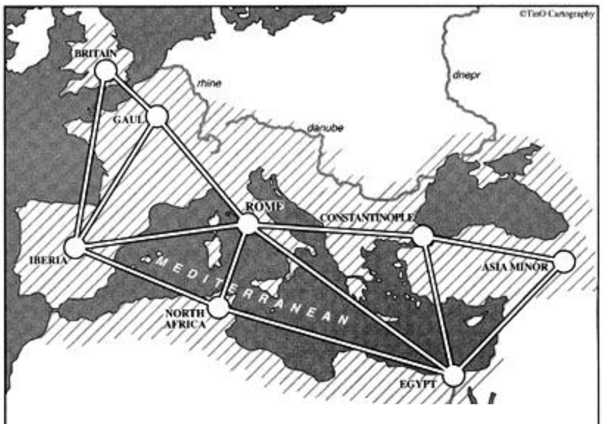

Figure 1: map showing the regions and the steps between the regions (courtesy of American Mathematics Association)

Indeed, there are six ways to improve on Constantine's deployment. These results are obtained through a form of zero-one integer programming by ReVelle & Rosing (2).

Besides placing of Roman army units, the same sort of math can also be used for optimizing the location of declining number of British Fleets at the end of 19th century or American Military Unit during the Cold War (2). In addition to army placement, the same sort of math is also useful when people want to know the best place in town to put a new hospital, fire station, or fast-food restaurant. Many times such optimization problems can be modeled by Roman domination or its variants.

2. Definitions and existing results

For basic definitions in graph theory, refer to Appendix 1.

Let $G = (V, E)$ be a graph. A **Roman dominating function** is a function $f : V \rightarrow \{0, 1, 2\}$ such that every vertex v for which $f(v) = 0$ has a neighbor u with $f(u) = 2$.

The **weight** of a Roman dominating function f is $w(f) = \sum_{v \in V} f(v)$, corresponding to the total number of army units required under a specific deployment scheme – a function f.

We are interested in finding Roman dominating function(s) of minimum weight for a particular graph; it makes sense: in the army placement context, we want to minimize the number of army units needed to secure a particular set of given regions.

This Roman dominating function(s) of minimum weight among all the possible Roman dominating functions is/are called (a) $\boldsymbol{\gamma_R}$**-function**(s). The **Roman domination number** of a graph G, denoted by $\gamma_R(G)$, is the weight of γ_R-function(s) – the minimum weight of all possible Roman dominating functions.

The following result is proven by Dreyer (3).

Proposition 1: For path P_n and cycle C_n of order n,

$$\gamma_R(P_n) = \gamma_R(C_n) = \left\lceil \frac{2n}{3} \right\rceil$$

where $\lceil x \rceil$ denotes the smallest integer larger than or equal to x.

The next observation follows readily from the definition.

Proposition 2: If H is a spanning subgraph of a graph G, then $\gamma_R(H) \geq \gamma_R(G)$.

3. Roman dominating index

Many times we are concerned with adding or deleting an edge and how it will affect the Roman domination number of a graph. In practice, armies, utility operators, etc are concerned about where to build a new road, a pipeline, etc so as to reduce the size of army or reap the most economic benefits. As such, I would like to introduce a new concept called Roman dominating index. It will be immensely useful in simpilying some Roman domination problems like the one shown in Section 3.2.

Let G be a graph and x, y two non-adjacent vertices in G. The **Roman dominating index** of $\{x, y\}$, denoted by $R(x, y)$, is defined by $R(xy) = \gamma_R(G) - \gamma_R(G + xy)$.

As G is a spanning subgraph of $(G + xy)$, by Proposition 2, $R(xy) \geq 0$. In what follows, we shall show that this quantity is always bounded above by 1.

3.1. The bounds of Roman dominating index

Proposition 3: Let G be a graph. For any pair of non-adjacent vertices $\{x, y\}$ in G, $0 \leq R(xy) \leq 1$.

Proof: We need only to prove that $R(xy) \leq 1$. Let $G' = G + xy$ and f' be a γ_R-function of G'. There are two cases to consider.

Case 1: $\{f'(x), f'(y)\} = \{0,2\}$.

WLOG, assume that $f'(x) = 0$ and $f'(y) = 2$, and define $f : V \rightarrow \{0, 1, 2\}$:

$$f(v) = \begin{cases} f'_{\gamma_R}(v), v \neq x \\ 1, v = x \end{cases}$$

Then f is a Roman dominating function of G as removing edge xy only raises the possibility that vertex x may be unprotected, if f'_{γ_R} for G' is to be used for G. Simply adding one more troop to this vertex x will resolve the issue – in this way, all vertices are again protected, with an increase of one in Roman domination number.

Clearly, $w(f)=\gamma_R(G^{'})+1$. Thus $\gamma_R(G)\leq w(f)=\gamma_R(G^{'})+1$. It follows that $R(xy)=\gamma_R(G)-\gamma_R(G^{'})\leq 1$.

Case 2: the negation of case 1.

As neither of the two vertices x and y is assigned to 2 by this function, the existence of the edge xy in G' does not help protecting either vertex x or y. Thus when edge xy is removed from G' to get G, $f^{'}$ is still a Roman dominating function for G. Hence $\gamma_R(G)\leq w(f^{'})=\gamma_R(G')$. So $R(xy)=\gamma_R(G)-\gamma_R(G^{'})\leq 0$.

Summing up aforementioned two cases of discussion on upper bound, we have $R(xy)\leq 1$. ■

Remark: Both the lower and upper bounds are reachable.

To show the upper bound is reachable, the simplest case is Roman dominating index of an edge which joins the two ends of a path of order three, P_3, together to form a cycle, C_3.

To show the lower bound is achievable, the Roman dominating index of any edge that connects two non-neighboring vertices in cycle C_4 is one.

Corollary: Let $\{x,y\}$ be a pair of non-adjacent vertices in a graph G. Then $R(xy)=1$ if and only if there exists a γ_R-function f of G such that $\{f(x),f(y)\}=\{1,2\}$.

Proof:

Sufficiency: Assume, WLOG, that $f(x)=1$ and $f(y)=2$ for G, then define f' on $G^{'}$:

$$f'(v)=\begin{cases} f(v), v\neq x \\ 0, v=x \end{cases}$$

f' is a Roman dominating function as x is protected by y in G'.

Now that $w(f^{'})=\gamma_R(G)-1$, we have $\gamma_R(G^{'})\leq w(f^{'})=\gamma_R(G)-1$. So $R(xy)=\gamma_R(G)-\gamma_R(G^{'})\geq 1$. As $R(xy)\leq 1$, we have $R(xy)=1$.

Necessity: Assume $R(xy)=1$. As shown in the proof for Proposition 3, there exist a γ_R-function f' of G' such that

$$\{f'(x),f^{'}(y)\}=\{0,2\}$$

Assume $f'(x)=0, f'(y)=2$, then we have a Roman dominating function f for G as defined by

$$f(v)=\begin{cases} f'(v), v\neq x \\ 1, v=x \end{cases}$$

Note that $w(f) = \gamma_R(G^{'}) + 1$. Thus $w(f) = \gamma_R(G^{'}) + 1 = \gamma_R(G) - R(xy) + 1 = \gamma_R(G)$. By definition, f is a γ_R-function for G, with $f(x) = 1, f(y) = 2$. ■

3.2. An application of Proposition 3

Problem 1: Given a path P_n of order $n \geq 3$, are there pairs of non-adjacent vertices in P_n are such that $R(xy) = 1$? If yes, which pairs?

Solution: Let f_{γ_R} be a γ_R-function for P_n.

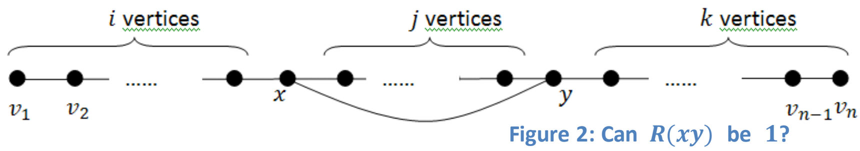

Figure 2: Can $R(xy)$ be 1?

If $n \equiv 0(\mathrm{mod}\ 3)$, no vertices in P_n are mapped to 1 in γ_R-function. According to the Corollary for Proposition 3, $R(xy) = 0$.

If $n \equiv 1(\mathrm{mod}\ 3)$, for non-trivial case where $n \neq 1$, there exists in γ_R-function for P_n a vertex mapped to 1 and vertices mapped to 2. Thus $\max\,[R(xy)] = 1$. To find the exact vertices to connect to obtain this maximum value, we just need to find the possible value-1 and value-2 vertices in f_{γ_R}. WLOG, let $f_{\gamma_R}(x) = 1$ and $f_{\gamma_R}(y) = 2$. As the positions of value-1 and value-2 vertices in f_{γ_R} follow a simple pattern, it is easy to show that Roman dominating index of 1 can be achieved iff $i \equiv 0, j \equiv 1, k \equiv 1\ (\mathrm{mod}\ 3)$.

Similarly, if $n \equiv 2(\mathrm{mod}\ 3)\ (n \neq 2)$, $\max\,[R(xy)] = 1$. WLOG assuming $f_{\gamma_R}(x) = 1$ and $f_{\gamma_R}(y) = 2$, the necessary and sufficient condition for which $R(xy) = 1$ is $i \equiv 0, j \equiv 1, k \equiv 2$ or $i \equiv 1, j \equiv 1, k \equiv 1$ or $i \equiv 0, j \equiv 2, k \equiv 1\ (\mathrm{mod}\ 3)$. ■

Remark 1: By similar argument, we can show that the maximum Roman dominating index and the required condition to achieve this extreme value for some other classes of graphs.

For cycle C_n of order n, if $n \equiv 0(\mathrm{mod} 3)$, $R(xy) = 0$. If $n \equiv 1\ or\ 2(\mathrm{mod}\ 3)$, $\max\,[R(xy)] = 1$.

For two disjoint paths/cycles, or a path and a cycle, if both of orders of the two components are not multiples of three, by joining the two disjoint components, we can have a Roman dominating edge with $R(xy) = 1$.

The exact positions of the vertices to connect can be determined by finding possible value-1 and value-2 vertices in f_{γ_R}, as discussed for P_n.

Remark 2: Without using Proposition 3, it will be much more tedious to solve Problem 1. See Appendix 2 for the alternative solution.

3.3. Discussion on adding successive new edges to a path

Problem 2: Given a path P_n of order $n \geq 3$, a positive integer m with $m \leq n$, and a vertex v not in P_n, how to add m new edges to join v and m vertices in P_n so that the resulting graph G has the largest $\gamma_R(G)$? What is the value of this largest $\gamma_R(G)$? What about the smallest one?

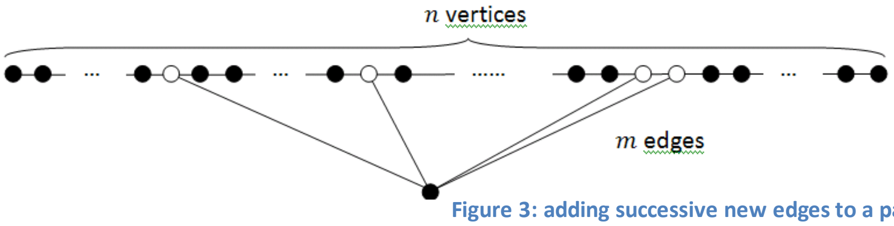

Figure 3: adding successive new edges to a path

Detailed solution is available in Appendix 3.

Result:

Largest: If $m \leq \left\lfloor \frac{n+1}{3} \right\rfloor + 1$, $\gamma_R(G) = \left\lceil \frac{2n+2}{3} \right\rceil$, and $f(v) = \begin{cases} 1, if\ n \equiv 0\ or\ 1\ (mod3) \\ 0, if\ n \equiv 2\ (mod3) \end{cases}$.

If $m \geq \left\lfloor \frac{n+1}{3} \right\rfloor + 1$, $\gamma_R(G) = n - m + 2$, and $f(v) = 2$.

Smallest: If $m \leq 3$, $\gamma_R(G) = \left\lceil \frac{2n}{3} \right\rceil$, and $f(v) = 0$.

If $m \geq 3$, $\gamma_R(G) = \left\lceil \frac{2}{3}(n-m) \right\rceil + 2$, and $f(v) = 2$.

Details on how to add the new edges to achieve these two extreme values are also available in Appendix 3. ■

Remark 1: This problem can model the transition from a segmented, line-like distribution system of gas/water/heat, to a centralized, star-like one.

Remark 2: Following the trend of adding an edge between two disjoint graph in sections 3.2 and adding successive edges in section 3.3, a direction for further research is to combine these two cases and look into the effect of adding successive edges between two disjoint graphs.

4. Bound of Roman domination number as a function of order

The **diameter** of a graph G, denoted by $D(G)$, is defined as

$$D(G) = \max\{d(u,v) | u, v \text{ are in } V\}.$$

where $d(u,v)$ denotes the distance between u and v in G.

Let $\lfloor a \rfloor$ denote the largest integer which is less than or equal to a real number a. Note that

$\lfloor a+b \rfloor \geq \lfloor a \rfloor + \lfloor b \rfloor$ for any real number a and b. We now establish the following result.

Proposition 4: For any tree T of order $n \geq 3$, $2 \leq \gamma_R(T) \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

Proof: The lower bound is trivial as no matter how large the order is, a star always has a Roman domination number of 2.

I will prove the upper bound by **mathematical induction** on the diameter of tree, $D(T)$.

Base cases: if $D(T) = 2, 3$, or 4, $\gamma_R(T) \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

Case 1: $D(T) = 2$. Obviously $\gamma_R(T) = 2 \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

Case 2: $D(T) = 3$. Find the path $v_0 e_0 v_1 e_1 v_2 e_2 v_3$ which maximizes $d(v_2)$.

If $d(v_1) > 2$ and $d(v_2) > 2$, we can remove e_1 and thus get two isolated trees T_1 and T_2 of diameter 2. $\gamma_R(T) \leq \gamma_R(T_1) + \gamma_R(T_2) \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

Otherwise, let $f(v_2) = 2$ and $f(v_0) = 1 \Rightarrow \gamma_R(T) = 3 \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

Case 3: $D(T) = 4$. Find a path $v_0 e_0 v_1 e_1 v_2 e_2 v_3 e_3 v_4$ which maximizes $d(v_3)$.

If $d(v_3) > 2$, we can remove it together with all its neighboring end vertices as a tree of diameter 2. Repeat this until the tree decreases in diameter to some cases previously discussed, or become a tree T' where $d(v_1) = d(v_3) = 2$ and $d(v_2) \geq 2$. The first case is handled by previously discussed trees of diameter 2 or 3. As to the second case, $f_{\gamma_R}(v_0) = f_{\gamma_R}(v_4) = 1$ and $f_{\gamma_R}(v_2) = d = 2$. Then we consider the worst case: a star-like structure where every end vertex is distance 2 away from v_2. Thus we have $\gamma_R(T') \leq \frac{2+d}{2d+1} n' \leq \left\lfloor \frac{4n'}{5} \right\rfloor$, where n' is the order of T'.

Inductive hypothesis: If Proposition 4 holds for any tree T of $k-3 \leq D(T) \leq k-1$, then it also holds for any tree T of $D(T) = k$. To show this:

1. Given a tree T of $D(T) = k$, find its longest path $v_0 e_0 v_1 e_1 v_2 \ldots v_{k-3} e_{k-3} v_{k-2} e_{k-2} v_{k-1} e_{k-1} v_k$.
2. Remove edge e_{k-3}. Since there is only one path linking a vertex to another in any tree, removing an edge means that these two vertices are no longer linked by edges or vertices. Thus two disjoint trees T_{b1} and T_1 result.

   T_{b1} contains path $v_{k-2} e_{k-2} v_{k-1} e_{k-1} v_k$. $d(v_{k-2}, v_k) = 2 \Rightarrow D(T_{b1}) \geq 2$. Since we chose the longest path in T, $D(T_{b1}) \leq 4$. Thus $2 \leq D(T_{b1}) \leq 4$ and T_{b1} falls in base cases aforementioned. Let

$v(T_{b1})$ denote the order of T_{b1}. We have $\gamma_R(T_{b1}) \leq \left\lfloor\frac{4v(T_{b1})}{5}\right\rfloor$.

T_1 contains path $v_0e_0v_1e_1v_2 \dots v_{k-4}e_{k-4}v_{k-3}$. $d(v_0, v_{k-3}) = k-3 \Rightarrow D(T_1) \geq k-3$. In addition, $D(T_1) \leq D(T) = k$. Thus $k-3 \leq D(T_1) \leq k$. If $D(T_1) = k$, do note that there are fewer paths of length k in T_1 than in T as path $v_0e_0v_1e_1v_2 \dots v_{k-3}e_{k-3}v_{k-2}e_{k-2}v_{k-1}e_{k-1}v_k$ and possibly others no longer exist in T_1.

3. If $D(T_1) = k$, repeat step 1 and 2. At i^{th} repeating of step 1 and 2, we divide T_{i-1} into T_{bi} and T_i. As the number of path of length k is finite and this number decreases each time we apply step 1 and 2, we are certain that after m repeats $D(T_m)$ will for the first time be smaller than k. So we have $k-3 \leq D(T_m) \leq k-1$.

Assume $\gamma_R(T_m) \leq \left\lfloor\frac{4v(T_m)}{5}\right\rfloor$ as in the inductive hypothesis, for T of $D(T) = k$ we have

$$\gamma_R(T) \leq \gamma_R\left(\sum_{i=1}^{m} T_{bi} + T_m\right) = \sum_{i=1}^{m} \gamma_R(T_{bi}) + \gamma_R(T_m) \leq \sum_{i=1}^{m} \left\lfloor\frac{4v(T_{bi})}{5}\right\rfloor + \left\lfloor\frac{4v(T_m)}{5}\right\rfloor$$
$$\leq \left\lfloor\frac{4\sum_1^m v(T_{bi}) + 4v(T_m)}{5}\right\rfloor = \left\lfloor\frac{4v(T)}{5}\right\rfloor \qquad ■$$

Remark 1: This bound is achievable by constructing trees of the following structures.

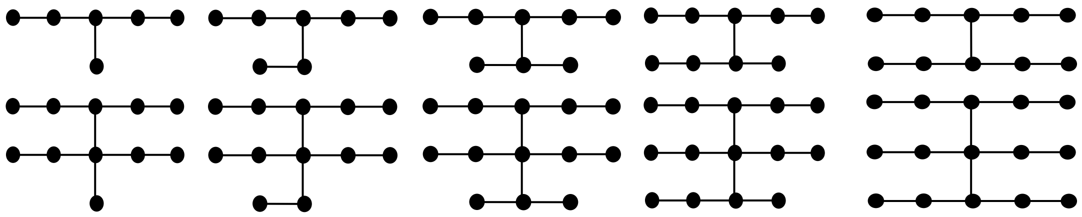

Figure 4

Remark 2: Given a tree of order $n \geq 3$, $\gamma_R(T) = \frac{4n}{5}$ if and only if T has a structure like the right most ones shown in Figure 4.

Proof: Sufficiency is shown directly by Proposition 4. For necessity, we need a closer examination of proof for Proposition 4. We find that given $2 \leq D(T) \leq 4$, only $\gamma_R(P_5) = \frac{4n}{5}$. Only when $T_{bi} = P_5$ for all $1 \leq i \leq m$ will we have $\gamma_R(T) = \frac{4n}{5}$. ■

Corollary: For any connected graph G of order $n \geq 3$, $2 \leq \gamma_R(G) \leq \left\lfloor\frac{4n}{5}\right\rfloor$.

Proof: Proof for lower bound is trivial while the one for upper bound follows immediately from Proposition 2 and 4. ■

5. Bibliography

1. **Stewart, Ian.** Defend the Roman Empire. *Scientific American.* December 1999, pp. 136-138.

2. **ReVelle, Charles S. and Rosing, Kenneth E.** Defendens Imperium Romanum: A Classical Problem in Military Strategy. *The American Mathematical Monthly, Vol. 107, No. 7.* Aug. - Sep. 2000, pp. 585-594.

3. **Dreyer, Paul Andrew.** *Center for Discrete Mathematics & Theoretical Computer Science.* [Online] October 2000. [Cited: December 15, 2008.] http://dimacs.rutgers.edu/~pdreyer/thesis.pdf.

4. **Koh, Khee Meng, Dong, Fengming and Tay, Eng Guan.** *Introduction to Graph Theory: H3 Mathematics.* Singapore : World Scientific Publishing Co. Pte. Ltd., 2007.

6. Appendices

6.1. Appendix 1: Definitions

Followings are some basic definitions in graph theory, many of which are adopted from *Introduction to Graph Theory: H3 Mathematics* (4).

A **graph** G consists of a non-empty finite set $V(G)$ of vertices together with a finite set $E(G)$ (possibly empty) of edges such that:

1. Each edge joins two *distinct* vertices in $V(G)$ and
2. Any two distinct vertices in $V(G)$ are joined by *at most one* edge.

The number of vertices in G, denoted by $v(G)$, is called the **order** of G.

Let u, v be any two vertices in G.

They are said to be **adjacent** if they are joined by an edge, say, e in G. We also write $e = uv$ or $e = vu$ (the ordering of u and v in the expression is immaterial), and we say that

1. u is a **neighbor** of v and vice versa,
2. the edge e is **incident with** the vertex u (and v) and
3. u and v are the two **ends** of e.

The set of all neighbors of v in G is denoted by $N(v)$; that is,

$$N(v) = \{x|x \text{ is a neighbor of } v\}.$$

The **degree** of v in G, denoted by $d(v)$, is defined as the number of edges incident with v. The vertex v is called an **end-vertex** if $d(v) = 1$.

A **path** in a graph G is an alternating sequence of vertices and edges beginning and ending at vertices:

$$v_0 e_0 v_1 e_1 v_2 \dots v_{k-1} e_{k-1} v_k$$

where $k \geq 1$, e_i is incident with v_i and v_{i+1}, for each $i = 0, 1, \dots, k-1$, and the vertices v_i's and edges e_i's need to be distinct. The **length** of the above path is defined as k, which is the number of occurrences of edges in the sequence.

A graph G is said to be **connected** if every two vertices in G are joined by a path, and **disconnected** if it is not connected.

The **distance** from u to v, denoted by $d(u, v)$, is defined as the *smallest length* of all $u - v$ paths in

G.

Let P_n denote a **path** of n vertices, $P_n = v_1v_2 \dots v_n$, and C_n a **cycle** of n vertices, $C_n = v_1v_2 \dots v_nv_1$.

Notice that we have two definitions for path. What a 'path' really means should be clear from the context when it is mentioned.

A graph H is called a **subgraph** of graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of a graph G is said to be **spanning** if $V(H) = V(G)$.

6.2. Appendix 2: Alternative solution for the problem in 3.2

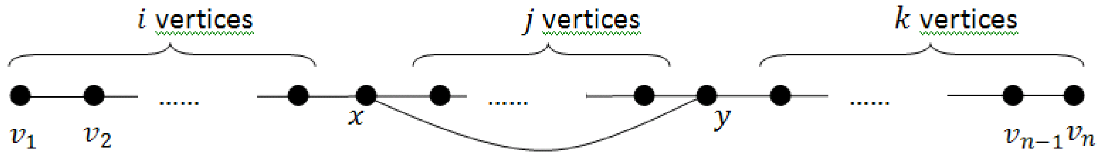

Figure 2: Can $R(xy)$ be 1?

The alternative solution:

We denote the path of order n by P_n, and the new graph formed by adding an extra edge by $P_n{}'$.

It is obvious that if neither vertices x nor y is assigned 2 under f'_{γ_R}, γ_R-function for $P_n{}'$, graph G and G' will have the same Roman domination number and thus the Roman dominating index of edge xy will always be zero. According to Proposition 1,

$$\gamma_R(P_n{}') = \left\lceil \frac{2n}{3} \right\rceil, f(x) \neq 2 \; and \; f(y) \neq 2$$

If one of vertices x and y is assigned 2 by function f'_{γ_R}. WLOG, let $f'_{\gamma_R}(x) = 0$ and $f'_{\gamma_R}(y) = 2$. As shown in Figure 1.1, i is the number of vertices on the left of x, j between x and y (not inclusive) and k on the right of y.

Numbers assigned to vertices x and y are already fixed (0 and 2 respectively). In addition, vertex y can protect its three neighbors in G'. The remaining is to find the Roman domination number for three paths, of order $i, (j-1), and\,(k-1)$, which can be easily done using Formula 1,

$$\gamma_R(P_n{}') = \left\lceil \frac{2i}{3} \right\rceil + \left\lceil \frac{2(j-1)}{3} \right\rceil + \left\lceil \frac{2(k-1)}{3} \right\rceil + 2, f(x) = 2 \text{ or } f(y) = 2$$

$$n = i + j + k + 2$$

Combining the two cases, we have

$$\gamma_R(P_n{}') = \min\left[\left\lceil \frac{2n}{3} \right\rceil, \left\lceil \frac{2i}{3} \right\rceil + \left\lceil \frac{2(j-1)}{3} \right\rceil + \left\lceil \frac{2(k-1)}{3} \right\rceil + 2\right],$$

where $n = i + j + k + 2$,

and $min[a, b]$ = the smaller value of a and b (if $a = b, min[a, b] = a = b$)

$$\therefore R(xy) = \left\lceil \frac{2n}{3} \right\rceil - \min\left[\left\lceil \frac{2n}{3} \right\rceil, \left\lceil \frac{2i}{3} \right\rceil + \left\lceil \frac{2(j-1)}{3} \right\rceil + \left\lceil \frac{2(k-1)}{3} \right\rceil + 2\right]$$

The following result can be easily checked:

no.	$i(mod 3)$	$j(mod 3)$	$k(mod 3)$	$R(xy)$

1	0	0	0	0
2	0	0	1	0
3	0	0	2	0
4	0	1	0	0
5	0	1	1	1
6	0	1	2	1
7	0	2	0	0
8	0	2	1	1
9	0	2	2	0
10	1	0	0	0
11	1	0	1	0
12	1	0	2	0
13	1	1	0	0
14	1	1	1	1
15	1	1	2	0
16	1	2	0	0
17	1	2	1	0
18	1	2	2	0
19	2	0	0	0
20	2	0	1	0
21	2	0	2	0
22	2	1	0	0
23	2	1	1	0
24	2	1	2	0
25	2	2	0	0
26	2	2	1	0
27	2	2	2	0

Thus the conclusion follows that:

For path P_n with order n,

If $n \equiv 0(mod3)$, $max[R(xy)] = 0$.

If $n \equiv 1(mod3), max\ [R(xy)] = 1$. Refer to combination No.5 for which two vertices to connect.

If $n \equiv 2(mod3), max\ [R(xy)] = 1$. Refer to combination No.6, No.8 and No.14 for which two vertices to connect. ■

Remark: Similar problems on other classes of graph as mentioned in Remark 1 of section 3.2 can also be solved in the same way whereby graphs of unknown Roman domination number is transformed to some classes of graphs of known Roman domination number such as a path.

6.3. Appendix 3: Detailed discussion on adding successive new edges

Problem 2: Given a path P_n of order $n \geq 3$, a positive integer m with $m \leq n$, and a vertex v not in P_n, how to add m new edges to join v and m vertices in P_n so that the resulting graph G has the largest $\gamma_R(G)$? What is the value of this largest $\gamma_R(G)$? What about the smallest one?

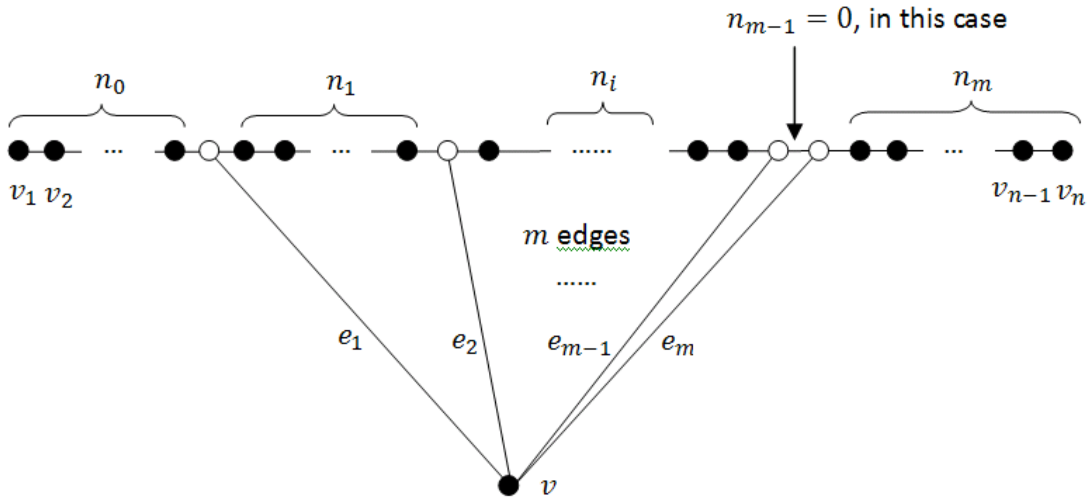

Figure 5: Adding successive new edges in detail

Solution:

Largest:

I would like to define two terms which will be used first.

A vertex v is **secured** with respect to f if $f(v) \neq 0$. A vertex v is **securable** with respect to f if $f(v) = 0$, but there exists vertex $u \in N(v)$ such that $f(u) = 2$.

I will divide the problem into two cases based on whether v is mapped to 2.

Case 1: v is mapped to 2. The case will be further divided into two sub-cases, based on whether the number of new edges reaches a threshold which makes it possible to eliminate three consecutive vertices which are not connected to v (i.e. all $n_i \leq 2$, for $0 \leq i \leq m$).

Specifically, the threshold value of the number of new edges is $\left\lfloor \frac{n}{3} \right\rfloor$, where $\lfloor x \rfloor$ is the floor function. In another word, when $m \geq \left\lfloor \frac{n}{3} \right\rfloor$, it is possible to achieve $n_i \leq 2$, for $0 \leq i \leq m$.

The "dispersing" algorithm is to choose $v_{3i}, 1 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor$ as the first $\left\lfloor \frac{n}{3} \right\rfloor$ vertices to connect to v. You can easily check that after these $\left\lfloor \frac{n}{3} \right\rfloor$ vertices are connected to v, the original path of order n is so “segmented” that there are no three or more consecutive vertices which do not connect to v.

Case 1.1: the threshold number of edges is reached. This case is straight-forward. By equally “dispersing” the vertices on the path which connect to the vertex v, it is possible to achieve $n_i \leq 2$, for $0 \leq i \leq m$.

As a result any n_i segment (as said, $n_i \leq 2$) will have a local Roman domination number equal to n_i. For example when $n_i = 2$, no matter whether you map 2 to one of the two vertices, or map 1 to both of the vertices, the local Roman domination number is always 2.

That being said, it is obvious that the maximum Roman domination number we can have when v is mapped to 2 and $m \geq \left\lfloor \frac{n}{3} \right\rfloor$ is

$$n - m + 2, (m \geq \left\lfloor \frac{n}{3} \right\rfloor)$$

where $(n - m)$ is the sum of the local Roman domination number for the vertices on the path which are not connected to v and 2 is the value v is mapped to.

Case 1.2: the threshold number of edges is not reached. I claim that there will always be some three (or more than three) consecutive vertices on the path which are not connected to v.

This can be easily shown.

Let $n = 3k_n + n^{'}$, where $n^{'} = 0, 1\ or\ 2$.

$$m < \left\lfloor \frac{n}{3} \right\rfloor = k_n$$

$$\Rightarrow 3m < 3k_n$$

As m and k_n are both integers,

$$\Rightarrow 3(m + 1) \leq 3k_n$$

$$\Rightarrow 3m + 2 < 3k_n \leq n\ (*)$$

For contradiction, if there is no three consecutive vertices which are not connected to v, the maximum order of the path we can have is

$$2(m + 1) + m = 3m + 2$$

We have $(m + 1)$ because there can be two-vertex segments at the two ends of the path and $(m - 1)$ in between.

In another word,

$$3m + 2 \geq n$$

This contradicts $(*)$.

It completes the proof that when $f(v) = 2, m < \left\lfloor \frac{n}{3} \right\rfloor$, there will always be some three (or more than three) consecutive vertices on the path which are not connected to v (i.e. $\exists n_i \geq 3$).

Now I will go on to the details on the case when the threshold value is not reached. I will show that when there is any three (or more) consecutive vertices on the path which are not connected to v, in order to maximize Roman domination number, it is always prudent to have all other "segments" be of order $(3k + 2)$.

In another word, if there exists $n_i \geq 3$, $n_j \equiv 2\ (mod 3)\ (for\, 0 \leq j \leq m, j \neq i)$ will always be among the Roman dominating functions of minimum weight, given $f(v) = 2, m < \left\lfloor \frac{n}{3} \right\rfloor$.

This can be shown by a simple algorithm. If there exist $n_j \not\equiv 2\ (mod 3)\ (j \neq i)$, we can "cut out" a vertex or two from the "segment" of order n_i and "insert" it into the "segment" of order n_j to achieve $n_j \equiv 2\ (mod 3)$. We can show that this algorithm will only increase the total Roman domination number for the graph or keep it constant.

In specific, if $n_j \equiv 0\ (mod 3)$, we can "cut out" 2 vertices from the segment of order n_i and insert it into the segment n_j. In doing so, the local Roman domination number for the segment n_j increases by 2, from $\frac{n_j}{3}$ to $(\frac{n_j}{3} + 2)$, while the local Roman domination number for the segment n_j decreases by equal to or less than 2 as $\left\lceil \frac{2n_i}{3} \right\rceil - \left\lceil \frac{2(n_i - 2)}{3} \right\rceil \leq 2$.

Similarly, by "cutting and pasting" 1 vertex, it can be shown that transforming any "segment" of order $(3k + 1)$ to $(3k + 2)$ never decreases the minimum weight of the graph.

Thus it is shown that given $f(v) = 2, m < \left\lfloor \frac{n}{3} \right\rfloor$, it never hurts to have as many $n_j \equiv 2\ (mod 3)$ as possible.

Thus one of the algorithms is to connect m edges to vertices $v_{3i}, 1 \leq i \leq m$.

In this way, the Roman domination number will be

$$2m + \left\lceil \frac{2}{3}(n - 3m) \right\rceil + 2, (m < \left\lfloor \frac{n}{3} \right\rfloor)$$

Simplify,

$$\left\lceil \frac{2}{3}n \right\rceil + 2, (m < \left\lfloor \frac{n}{3} \right\rfloor)$$

Summary of Case 1: the maximum weight achievable for the resultant graph formed by connecting a new vertex v by m edges to a path of order n, given $f(v)=2$, is

$$\begin{cases} \left\lceil \frac{2}{3}n \right\rceil + 2, (m < \left\lfloor \frac{n}{3} \right\rfloor) \\ n-m+2, (m \geq \left\lfloor \frac{n}{3} \right\rfloor) \end{cases}$$

Case 2: v is not mapped to 2. No vertices on the path is securable by v, thus the "local weight" for the path itself is $\left\lceil \frac{2n}{3} \right\rceil$. However, it remains a question whether it is always possible to "force" v to be mapped to 1, by cleverly choosing vertices on the path to connect.

A closer inspection reveals that in this case we need to have a discussion based on the order of path modulus 3.

Case 2.1: When $n=3k_n$, the minimum local weight for the path is $2k_n$. We have no choice but to map vertices $v_{3i-1}\ (1 \leq i \leq k_n)$ to 2 while others on the path to 0.

When $m \leq 2k_n$, we connect the new edges to vertices others than $v_{3i-1}\ (1 \leq i \leq k_n)$. v is not securable and must be mapped to 1. Thus the minimum weight possible is

$$\left\lceil \frac{2n}{3} \right\rceil + 1, m \leq 2k_n = \frac{2n}{3}$$

When $m > 2k_n$, we are forced to connect the new edges to some vertex (vertices) from $v_{3i-1}\ (1 \leq i \leq k_n)$. v is thus securable and will be mapped to 0 under the Roman domination function of minimum weight. Thus the minimum weight possible is

$$\left\lceil \frac{2n}{3} \right\rceil, m > 2k_n = \frac{2n}{3}$$

In summary, when v is not mapped to 2 and $n \equiv 0\ (mod\ 3)$, minimum weight achievable is

$$\begin{cases} \left\lceil \frac{2n}{3} \right\rceil + 1 = \frac{2n+3}{3} = 2k_n + 1, m \leq \frac{2n}{3} = 2k_n \\ \left\lceil \frac{2n}{3} \right\rceil = \frac{2n}{3} = 2k_n, m > \frac{2n}{3} = 2k_n \end{cases}$$

Case 2.2: When $n = 3k_n + 1$, the minimum local weight for the path is $(2k_n+1)$. In the path, there are k_n 3-vertex "segments" which are securable by its central vertex which is mapped to 2, plus an additional vertex which is mapped to 1 by itself.

Unlike the case when $n=3k_n$, we have some rooms for maneuver in this case. If $f(v_n)=1$, vertices $v_{3i-1}\ (1 \leq i \leq k_n)$ will be mapped to 2 while others on the path to 0. If $f(v_1)=1$, vertices $v_{3i}\ (1 \leq$

$i \leq k_n)$ will be mapped to 2 while others on the path to 0. Only vertices $v_{3i-2}\ (1 \leq i \leq k_n + 1)$ will never be mapped to 2 if minimum local weight for the path, $(2k_n + 1)$ is to be achieved.

Thus when $m \leq k_n + 1$, we connect the new edges to vertices $v_{3i-2}\ (1 \leq i \leq k_n + 1)$. v is not securable and must be mapped to 1. Thus the minimum weight possible is

$$\left\lceil \frac{2n}{3} \right\rceil + 1, m \leq k_n + 1$$

When $m > k_n + 1$, we are forced to connect the new edges to some vertex (vertices) from $v_{3i-1}\ (1 \leq i \leq k_n)$ and $v_{3i}\ (1 \leq i \leq k_n)$, which can be mapped to 2 without increase in the local minimum weight for the path. v is thus securable and will be mapped to 0 under the Roman domination function of minimum weight. Thus the minimum weight possible is

$$\left\lceil \frac{2n}{3} \right\rceil, m > k_n + 1$$

In summary, when v is not mapped to 2 and $n \equiv 1 (mod\ 3)$, minimum weight achievable is

$$\begin{cases} \left\lceil \frac{2n}{3} \right\rceil + 1 = \frac{2n+4}{3} = 2k_n + 2, m \leq k_n + 1 = \frac{n+2}{3} \\ \left\lceil \frac{2n}{3} \right\rceil = \frac{2n+1}{3} = 2k_n + 1, m > k_n + 1 = \frac{n+2}{3} \end{cases}$$

Case 2.3: when $n = 3k_n + 2$, the minimum local weight for the path is $(2k_n + 2)$. In the path, there are k_n 3-vertex "segments" which are securable by its central vertex which is mapped to 2, plus two additional vertices which can either be mapped to 1 by themselves or can been seen as a 2-vertex segment (one of the vertices mapped to 2).

Now we have even more rooms for maneuver then in the previous case, and it turns out that any vertex on the path may be mapped to 2 and at the same time the local minimum weight for the path is achieved. If $f(v_{n-1}) = f(v_n) = 1$, vertices $v_{3i-1}\ (1 \leq i \leq k_n)$ will be mapped to 2 while others on the path to 0. If $f(v_1) = f(v_n) = 1$, vertices $v_{3i}\ (1 \leq i \leq k_n)$ will be mapped to 2 while others on the path to 0. If $f(v_1) = f(v_2) = 1$, vertices $v_{3i+1}\ (1 \leq i \leq k_n)$ will be mapped to 2 while others on the path to 0.

v_1 can also be mapped to 2: $f(v_1) = 2, f(v_2) = 0, f(v_{3i+1}) = 2\ (1 \leq i \leq k_n)$, all other vertices being mapped to 0. By symmetry, v_n can also be mapped to 2. Note that paths of order $(3k_n + 2)$ and $(3k_n + 3)$ are indeed the same in Roman domination number.

As a conclusion, when v is not mapped to 2 and $n \equiv 2\ (mod\ 3)$, the minimum weight achievable is always

$$\left\lceil\frac{2n}{3}\right\rceil = \frac{2n+2}{3} = 2k_n + 2$$

Now we have obtained the minimum weight possible under each cases depending on whether $f(v) = 2$, we can compare which case offers the smaller value and under which conditions. We will be comparing case by case, based on modulus 3 and will simplify any result if possible.

Case 3.1: When $n \equiv 0\ (\mathrm{mod}\ 3)$ or in another word $n = 3k_n$, Roman domination number for the resultant graph is

$$\min\left[\begin{cases} \frac{2n}{3}+2, m < \frac{n}{3} \\ n-m+2, m \geq \frac{n}{3} \end{cases}, \begin{cases} \frac{2n}{3}+1, m \leq \frac{2n}{3} \\ \frac{2n}{3}, m > \frac{2n}{3} \end{cases}\right]$$

Simplify,

When $0 \leq m < \frac{n}{3}$, $\gamma_R(G) = \frac{2n}{3} + 1$, v mapped to 1.

When $\frac{n}{3} \leq m \leq \frac{2n}{3}$, we have when $m \leq \frac{n}{3} + 1$, $n - m + 2 \geq \frac{2n}{3} + 1$ and when $m \geq \frac{n}{3} + 1$, $n - m + 2 \leq \frac{2n}{3} + 1$.

We notice that to satisfy,

$$\begin{cases} m \geq \frac{n}{3} + 1 \\ m \leq \frac{2n}{3} \end{cases}$$

$$\Rightarrow n \geq 3$$

For $n \equiv 0\ (mod\ 3)$ and $n > 0$, $n \geq 3$ is a trivial condition of no concern.

Similarly, we also check that

$$\frac{n}{3} \leq m \leq \frac{n}{3} + 1$$

always holds.

Thus we have the following result.

When $\frac{n}{3} \leq m \leq \frac{n}{3} + 1$, $\gamma_R(G) = \frac{2n}{3} + 1$, v mapped to 1.

When $\frac{n}{3} + 1 \leq m \leq \frac{2n}{3}$, $\gamma_R(G) = n - m + 2$, v mapped to 2.

When $\frac{2n}{3} < m \leq n$, we have when $m \leq \frac{n}{3} + 2$, $n - m + 2 \geq \frac{2n}{3}$ and when $m \geq \frac{n}{3} + 2$, $n - m + 2 \leq \frac{2n}{3}$.

We notice that to satisfy,

$$\begin{cases} \frac{2n}{3} < m \\ m \leq \frac{n}{3} + 2 \end{cases}$$

$$\Rightarrow n < 6 \Rightarrow n \leq 3$$

While to satisfy,

$$\begin{cases} m \leq n \\ m \geq \frac{n}{3} + 2 \end{cases}$$

$$\Rightarrow n \geq 3$$

As discussed in the case when $\frac{n}{3} \leq m \leq \frac{n}{3} + 1$, it is inherently assumed in the question that $n \geq 3$. In addition, it can be checked that when $n = 3$, $m = 3$ and the case of $m \leq \frac{n}{3} + 2$ can be encompassed in the case of $m \geq \frac{n}{3} + 2$.

Thus we have the following result.

When $\frac{2n}{3} < m \leq n$, $\gamma_R(G) = n - m + 2$, v mapped to 2.

To conclude for the case where $n \equiv 0\ (mod\ 3)$,

When $0 \leq m \leq \frac{n}{3} + 1$, $\gamma_R(G) = \frac{2n}{3} + 1$, v mapped to 1.

When $\frac{n}{3} + 1 \leq m \leq n, \gamma_R(G) = n - m + 2$, v mapped to 2.

Case 3.2: When $n \equiv 1\ (\mathrm{mod}\ 3)$ or in another word $n = 3k_n + 1$, Roman domination number for the resultant graph is

$$\min\left[\begin{cases} \frac{2n+1}{3} + 2, m < \frac{n-1}{3} \\ n - m + 2, m \geq \frac{n-1}{3} \end{cases}, \begin{cases} \frac{2n+4}{3}, m \leq \frac{n+2}{3} \\ \frac{2n+1}{3}, m > \frac{n+2}{3} \end{cases}\right]$$

Simplify,

When $0 \le m < \frac{n-1}{3}$, $\gamma_R(G) = \frac{2n+4}{3}$, v mapped to 1.

When $\frac{n-1}{3} \le m \le \frac{n+2}{3}$, we have only two possible integer values for m:

When $m = \frac{n-1}{3}$, $n - m + 2 = \frac{2}{3}n + \frac{7}{3} > \frac{2n+4}{3}$,

When $m = \frac{n+2}{3}$, $n - m + 2 = \frac{2}{3}n + \frac{4}{3} = \frac{2n+4}{3}$.

Thus when $\frac{n-1}{3} \le m \le \frac{n+2}{3}$, $\gamma_R(G) = \frac{2n+4}{3}$, v mapped to 1. However, do note that when $m = \frac{n+2}{3}$, $\gamma_R(G) = n - m + 2 = \frac{2n+4}{3}$.

When $\frac{n+2}{3} < m \le n$, we have $n - m + 2 \le \frac{2n+1}{3}$, with equality holds only when $m = \frac{n+5}{3}$. Thus $\gamma_R(G) = n - m + 2$, v mapped to 2. Do not that when $m = \frac{n+5}{3}$, $\gamma_R(G) = n - m + 2 = \frac{2n+1}{3}$.

To conclude for the case where $n \equiv 1\ (mod\ 3)$,

When $0 \le m \le \frac{n+2}{3}$, $\gamma_R(G) = \frac{2n+4}{3}$, v mapped to 1.

When $\frac{n+2}{3} \le m \le n$, $\gamma_R(G) = n - m + 2$, v mapped to 2.

Case 3.3: When $n \equiv 2\ (\text{mod } 3)$ or in another word $n = 3k_n + 2$, Roman domination number for the resultant graph is

$$\min\left[\begin{cases}\dfrac{2n+2}{3} + 2, m < \dfrac{n-2}{3} \\ n - m + 2, m \ge \dfrac{n-2}{3}\end{cases}, \frac{2n+2}{3}\right]$$

Simplify,

When $0 \le m < \frac{n-2}{3}$, $\gamma_R(G) = \frac{2n+2}{3}$, v mapped to 0.

When $\frac{n-2}{3} \le m \le \frac{n+4}{3}$, $n - m + 2 \ge \frac{2n+2}{3}$, $\gamma_R(G) = \frac{2n+2}{3}$, v mapped to 0.

When $\frac{n+4}{3} \le m \le n$, $n - m + 2 \le \frac{2n+2}{3}$, $\gamma_R(G) = n - m + 2$, v mapped to 2.

Result: Summarizing the three cases,

If $m \le \left\lfloor \frac{n+1}{3} \right\rfloor + 1$,

$$\gamma_R(G) = \left\lceil \frac{2n+2}{3} \right\rceil,$$

$$f(v) = \begin{cases} 1, if\ n \equiv 0\ or\ 1\ (mod 3) \\ 0, if\ n \equiv 2\ (mod 3) \end{cases}$$

If $m \geq \left\lfloor \frac{n+1}{3} \right\rfloor + 1$,

$$\gamma_R(G) = n - m + 2,$$

$$f(v) = 2$$

This gives the maximum Roman domination number when a path of order n is connected to a vertex by m edges.

Smallest:

To find the minimum Roman domination number, we just need to reverse everything.

We still have two cases.

1. If v is mapped to 2. We ought to have as many $n_i \equiv 0\ (mod\ 3)$ as possible.

   One simple way to do that is to connect e_1 to the first vertex on the path, e_2 the second, and so on.

   The Roman domination number is

$$\left\lceil \frac{2}{3}(n-m) \right\rceil + 2$$

2. If v is not mapped to 2. It is obvious that it is always possible that v is mapped to 0 yet still securable in one move. Thus the Roman domination number is

$$\left\lceil \frac{2n}{3} \right\rceil$$

Combining two cases, the minimum Roman domination number is

$$min\left[\left\lceil \frac{2}{3}(n-m) \right\rceil + 2, \left\lceil \frac{2n}{3} \right\rceil\right]$$

By property of ceiling function, $\left\lceil \frac{2}{3}(n-m) \right\rceil + 2 = \left\lceil \frac{2}{3}(n-m) + 2 \right\rceil = \left\lceil \frac{2}{3}n + (2 - \frac{2}{3}m) \right\rceil$. Through this we compare the two terms easily.

The result is:

If $m \leq 3$,

$$\gamma_R(G) = \left\lceil \frac{2n}{3} \right\rceil$$

$$f(v) = 0$$

If $m \geq 3$,

$$\gamma_R(G) = \left\lceil \frac{2}{3}(n-m) \right\rceil + 2$$

$$f(v) = 2 \quad \blacksquare$$