INVESTIGATING WAYS TO VISUALIZE AND DEFINE LENS SPACES

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SCIENCE RESEARCH PROGRAMME

Abstract

The paper deals with the thorough study of three models of lens spaces, their elementary properties and provides proofs that the three models are equivalent.

Introduction

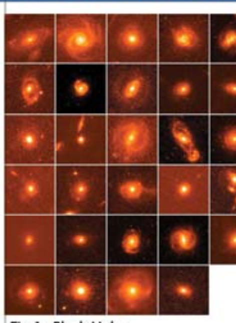


Fig 1: Black Holes
Image taken from NASA/JPL-Caltech/NOAO/AURA/NSF

In three spatial dimensions, event horizons of black holes are proven to be spherical in shape and not any other shape. This was proven mathematically and considered the uniqueness theorem. However, in 4 and higher spatial dimensions, it has been proven mathematically that other shapes of event horizon do exist - the uniqueness theorem does not apply. These shapes are described by lens spaces. However, there is an inherent difficulty for physicists to visualise and comprehend these spaces. The research will therefore aims to analyse and explain lens spaces in a visually clear and concise manner, facilitating further development of the topic.

Preliminary Definitions

A torus knot is an imaginary knot which lies on the surface of an unknotted torus. The torus knot is described using latitudes and meridians which are specified by a pair of integers (p, q) in which gcd(p, q) =1 and $0 \leq q < p$. The (p, q) torus knot winds p times along the latitudes while at the same time winds q times along the meridians.

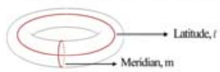


Fig 2: Latitude and meridian axises

Fig 3: (2, 1) torus knot

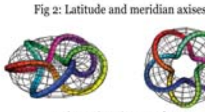


Fig 4: (2, 5) torus knot

Fig 5: (5, 3) torus knot

*Image generated from Java View, http://www.javaview.de/vgp/tutor/torusknot/PaTorusKnot.html

L(p, q) denotes a lens space. There are many possible lens spaces that satisfy the condition stated above. i.e. L(2,1), L(3,2), L(5,3), L(7,3) etc.

3 Models of Lens Spaces

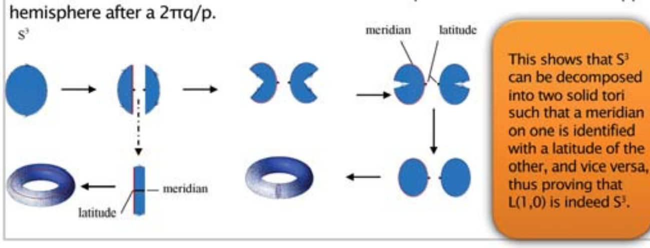
Model 1

The first model of L(p, q) says that a lens space is formed when a solid lens-shaped cell whose surface consists of two identical, radially symmetric caps which meet at a circular rim, L(p, q) is formed from the identification of upper and lower caps via an orthogonal projection after a 2πq/p positive rotation of the upper cap with respect to the lower.

Fig 6: representation of model 1

Model 2

The second model describes L(p, q) as the result of joining two solid tori T_1, T_2 via a homeomorphism h: $\partial T_1 \rightarrow \partial T_2$ where h takes a meridian m on ∂T_1 to a torus knot (p, q) on ∂T_2. The effect can be seen from the special case of lens spaces: L(1, 0) = S^3. L(1, 0) is the result of homeomorphism of the meridian m on ∂T_1 to torus knot (1, 0) on ∂T_2 which is the latitude. The resultant lens space is S^3. From model 1, S^3 is formed from identification of surface of lower hemisphere with the surface of upper hemisphere after a 2πq/p.



Model 3

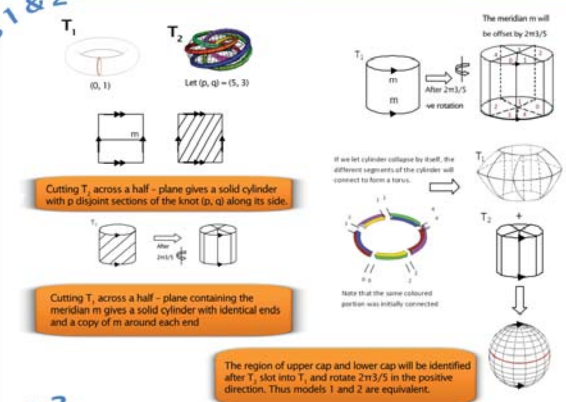
The third model of lens spaces defines L(p, q) as resultant space of identification by a Z_p action (Z_p is a finite cyclic group) on the space S^3. It is denoted as S^3/Z_p.

Fig 8: Simplified model of S^3 using complex numbers

$Z_p = \{0,1,\ldots,p-1\}$ is defined as a cyclic group with p elements where addition is defined mod p. Let $q \in Z_p$ with gcd(p, q) =1 and $0 \leq q < p$. Let Z_p act on S^3 as follows: $m \cdot (z_0, z_1) = (e^{2\pi i m/p} \cdot z_0, e^{2\pi i q m/p} \cdot z_1)$. This means that it will rotate z_0, z_1 by certain angles.

Equivalence of the 3 Models

Models 1 & 2



Models 1 & 3

From model 3, S^3 can be the set of complex numbers (z_0, z_1), where $z_0 = r_0 e^{i\theta_0}$, $z_1 = r_1 e^{i\theta_1}$ and $r_0^2 + r_1^2 = 1$. Further fixing $r_0 = (1 - r_1^2)^{1/2}$, (z_0, z_1) can be associated with three variables, $(r_1, \theta_0, \theta_1)$. The three variables can be combined to form a solid torus T.

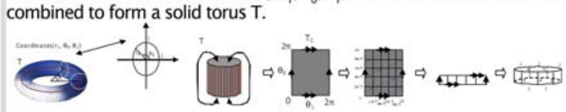


Fig 9

Consider S^3 under the Z_p action, $m \cdot (z_0, z_1) = (e^{2\pi i m/p} \cdot z_0, e^{2\pi i q m/p} \cdot z_1) = (r_0 \cdot e^{i(\theta_0 + 2\pi m/p)}, r_1 \cdot e^{i(\theta_1 + 2\pi m/p)})$. Therefore the Z_p action on T is $m \cdot (r, \theta_0, \theta_1) = (r, \theta_0 + 2\pi m/p, \theta_1 + 2\pi m q/p)$. r is unaffected by the Z_p action. Tc is individual tori T_c, where r = c ($0 \leq c \leq 1$). $T = \bigcup T_c$ When $0 < c < 1$, T_c can be represented as a square with identified edges, with the horizontal axis = θ_1-axis and the vertical axis = θ_0-axis. The action of m on the point (θ_0, θ_1) results in the point $(\theta_0 + 2\pi m/p, \theta_1 + 2\pi m q/p)$. Thus Z_p actions are translations along lines with gradient 1/q. This will result in the fundamental domain which is a short tube with ends identified after a 2πq/p rotation.

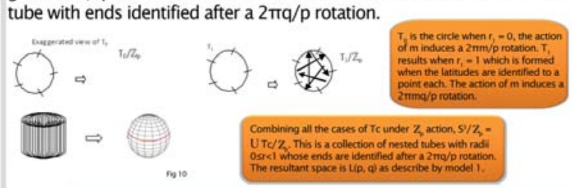


Fig 10

Conclusion

Combining the above results, the three models of lens spaces are consequently equivalent.

References

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