SRP SUBMISSION

INVESTIGATING WAYS TO VISUALIZE AND DEFINE LENS SPACES

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Investigating ways to visualize and define lens spaces

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1. Abstract

The paper deals with the thorough study of three models of lens spaces, their elementary properties and provides proofs that the three models are equivalent.

2. Introduction

The Lens space is an example of a topological space used in mathematics. It has many applications[1], one of which is to prove if a uniqueness theorem[2] applies to the event horizon of a black hole in higher spatial dimensions. In three spatial dimensions, event horizons of black holes are proven to be spherical in shape and not any other shape. This was proven mathematically and called the uniqueness theorem. However, in four and higher spatial dimensions, it has been proven mathematically that other shapes of the event horizon do exist – the uniqueness theorem does not apply! These shapes are described by lens spaces. However, there is an inherent difficulty for physicists to visualize and comprehend these higher dimensional spaces thoroughly. This paper analyses and explains lens spaces in a visually clear and concise manner, facilitating further development of the topic; our treatment will follow Watkins (1989).

3. Preliminary Definitions

3.1 Two topological spaces X and Y are called **homeomorphic** or topologically equivalent if there exists a bijective (i.e. one-one, onto) function f: $X \rightarrow Y$ such that f and f^{-1} are continuous. The function f is called a **homeomorphism**. (Lipschutz, 1965)

3.2 An **n-manifold** is a mathematical space in which every point has a neighbourhood which resembles Euclidean space[3].

For example, in a one-manifold, every point has a neighborhood that looks like a segment of a line. In a two-manifold, every point has a neighborhood that looks like a

[1] Other applications of lens spaces include construction of the Poincare Homology Sphere, Milnor's counterexample to the Hauptvermutung, study of homology theory, Whitehead torsion, K-theory etc.
[2] Uniqueness theorem: All objects of a given class are equivalent and can be represented by the same model.
[3] An n-dimensional space with notions of distance and angle that obey Euclidean relationships is called an n-dimensional Euclidean space.

disk. Note that in the study of lens spaces, we are only interested in the surface of the lens spaces.

3.3 A **torus knot** is an imaginary knot which lies on the surface of an unknotted torus. The torus knot is described using latitudes and meridians (ref fig 1) which are specified by a pair of integers (p, q) in which gcd(p, q) =1 and 0≤q<p. The (p, q) torus knot winds p times along the latitudes while at the same time winds q times along the meridians. Thus, the torus knot is link.

Fig 1.1: Latitude, l

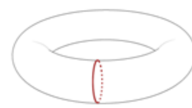


Fig 1.2: Meridian, m

*Fig 1.1 and 1.2 is also (p, 0) and (0, q) torus knot respectively.

Examples of (p, q) torus knots, where gcd(p, q) =1 and 0≤q<p. (Diagrams generated from reference 3)

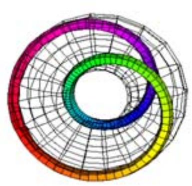


Fig 2.1: Top view of a (2, 1) torus knot

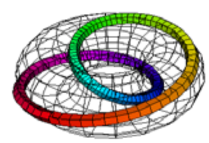


Fig 2.2: Side view of a (2, 1) torus knot

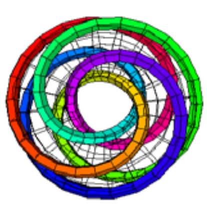


Fig 2.3: Top view of a (5, 3) torus knot

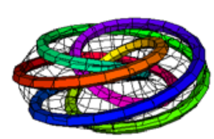


Fig 2.4: Side view of a (5, 3) torus knot

3.4 L(p, q) denotes a lens space. Thus, there are many possible lens spaces that satisfy the condition stated in 3.3. i.e. L(2,1), L(3,2), L(5,3), L(7,3) etc.

3.5 S^2: 2-dimensional surface of a sphere T^2: 2-dimensional surface of a torus

S^3: 3-dimensional **surface** of a sphere T^3: 3-dimensional **surface** of a torus

4. Models of the 3-Dimensional Lens Spaces

4.1 The first model of L(p, q) says that a lens space is formed when a solid lens-shaped cell whose surface consists of two identical, radially symmetric caps which meet at a circular rim, L(p, q) is formed from the identification of upper and lower caps via an orthogonal projection after a $2\pi q/p$ positive rotation of the upper cap with respect to the lower.

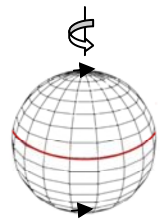


Fig 3

4.1.1 Elementary properties of model 4.1

L(0, q) or T^3 is not a lens space in this definition, because when p is 0, $2\pi q/0$ is undefined. L(1, q) is equal to S^3, since p is always 1 and no matter what the value of q is, it will always complete $2\pi q$ rounds.

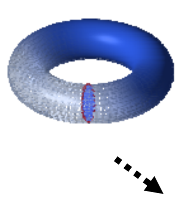
L(p, q) = L(p, q + mp), where m is an integer, since $2\pi(q + mp)/p = 2\pi q/p + 2\pi m$ and the effect of the rotation will not be affected by the addition of $2\pi m$.

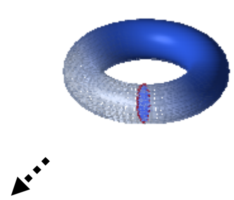
4.2 The second model describes L(p, q) as the result of joining two solid tori T_1, T_2 via a homeomorphism h: $\partial T_1 \rightarrow \partial T_2$ where h takes a meridian m on ∂T_1 to a torus knot (p, q) on ∂T_2.

Let's apply this definition to understand special case 1 of lens spaces: L(0, 1). We will see that it is equal to $\mathbf{T^3}$.

Fig 4.1: Solid T_1 with meridian m on its boundary

Fig 4.2: Torus knot (0, 1) on Solid T_2





The boundary of the discs from T_1 is mapped onto the boundary of discs from T_2. There is no mapping between the volumes of the two solid tori.

Imagine sewing up only the boundary of the two solid discs. What you get will be a sphere.

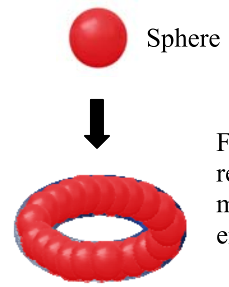


Fig 4.3: Hypothetical resultant torus, $\mathbf{T^3}$ which is made up of many spheres embedding on each other.

Next, let's apply this definition to special case 2 of lens space: L(1,0) and prove that it is equal to S^3. In this example, it is harder to visualize it using the previous method. We shall work backward to visualize the whole process.

Fig 4.4: Solid T_1 with meridian m on its boundary

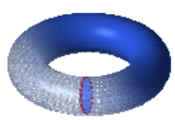
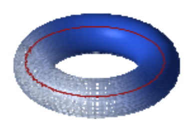


Fig 4.5: Torus knot (1, 0) on Solid T_2



Based on the definition 4.2, we know that meridian m on ∂T_1 is to be mapped to the torus knot (1, 0) on ∂T_2 which is the latitude. The resultant lens space is S^3. Based on definition 4.1, S^3 is formed from identification of surface of lower hemisphere with the surface of upper hemisphere after a $2\pi q/p$ (which is 0 rotation in this special case) positive rotation of the upper cap with respect to the lower.

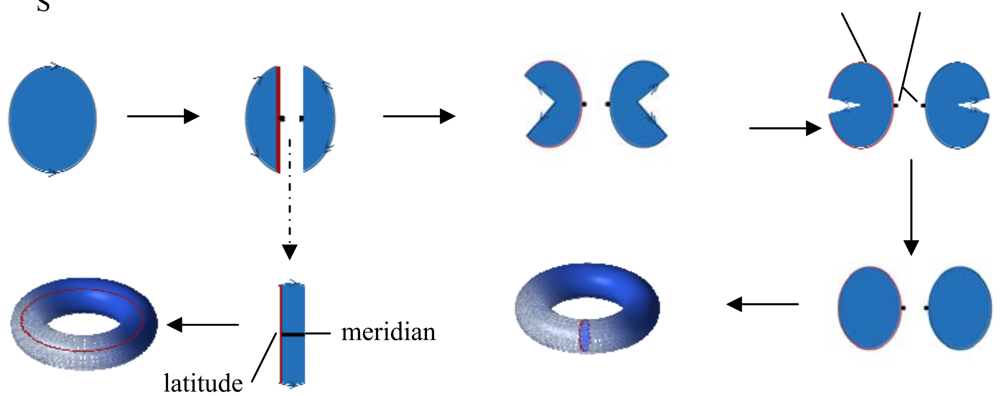


Fig 4.6: Cross-section diagram of the surgery process

This shows that S^3 can be decomposed into two solid tori such that a meridian on one is identified with a latitude of the other, and vice versa, thus proving that L(1,0) is indeed S^3.

4.2.1 Elementary properties of model 4.2

$L(0, q) = T^3$

4.3 The third model of lens spaces defines $L(p, q)$ as resultant space of identification by a $\mathbb{Z}_p$ action ($\mathbb{Z}_p$ is a finite cyclic group) on the space S^3. It is denoted as $S^3/\mathbb{Z}_p$.

To understand this, we will first construct an easily visualized model of S^3. Let S^3 be the set of complex numbers (z_0, z_1), where $z_0 = r_0 e^{i\theta_0}$**,** $z_1 = r_1 e^{i\theta_1}$ and $r_0^2 + r_1^2 = 1$**.**

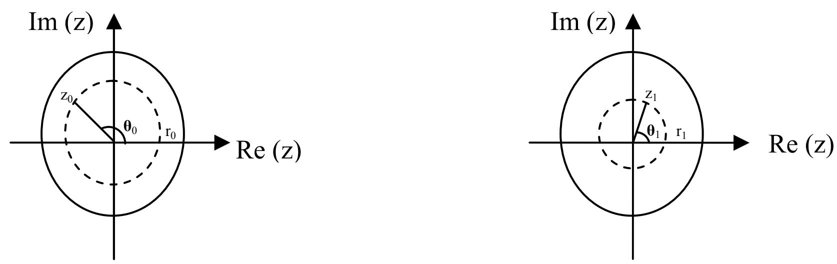
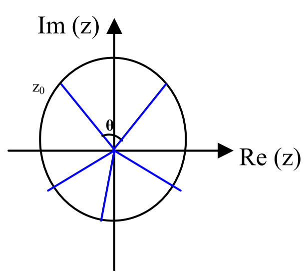


Fig 4.7: Simplified model of S^3 using complex numbers

Next, $\mathbb{Z}_p = \{0,1,\ldots,p-1\}$ is defined as a cyclic group with p elements where addition is defined mod p. Let $q \in \mathbb{Z}_p$ with gcd(p, q) =1 and $0 \le q < p$. Let $\mathbb{Z}_p$ act on S^3 as follows: $m \cdot (z_0, z_1) = (e^{2\pi i m/p} \cdot z_0, e^{2\pi i q m/p} \cdot z_1)$. This means that it will rotate z_0, z_1 by certain angles. Let's consider the case of L(5,3) to understand this definition better. $\mathbb{Z}_5 = \{0,1,2,3,4\}$ and S^3 is defined as the set of complex numbers (z_0, z_1).



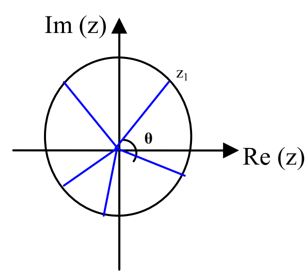


Fig 4.8

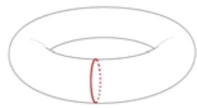
m = 0	$z_0 \rightarrow z_0$	m = 0	$z_1 \rightarrow z_1$
m = 1	$z_0 \rightarrow e^{2\pi i/5} \cdot z_0$	m = 1	$z_1 \rightarrow e^{2\pi i3/5} \cdot z_1$
m = 2	$z_0 \rightarrow e^{2\pi i2/5} \cdot z_0$	m = 2	$z_1 \rightarrow e^{2\pi i1/5} \cdot z_1$
m = 3	$z_0 \rightarrow e^{2\pi i3/5} \cdot z_0$	m = 3	$z_1 \rightarrow e^{2\pi i4/5} \cdot z_1$

m = 4 $z_0 \rightarrow e^{2\pi i4/5}\cdot z_0$ m = 4 $z_1 \rightarrow e^{2\pi i2/5}\cdot z_1$

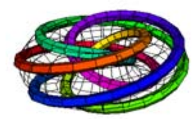
5 Link between definitions 4.1 and 4.2

Consider the two solid tori from definition 4.2.

T_1

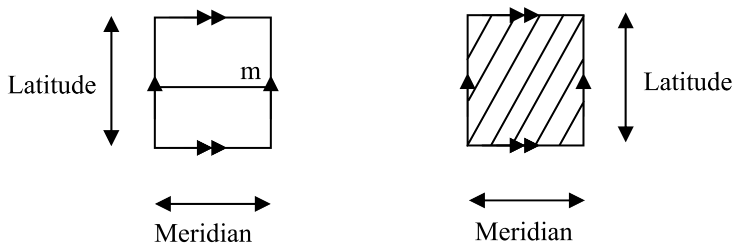


(0, 1)

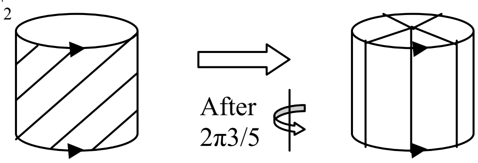


T_2

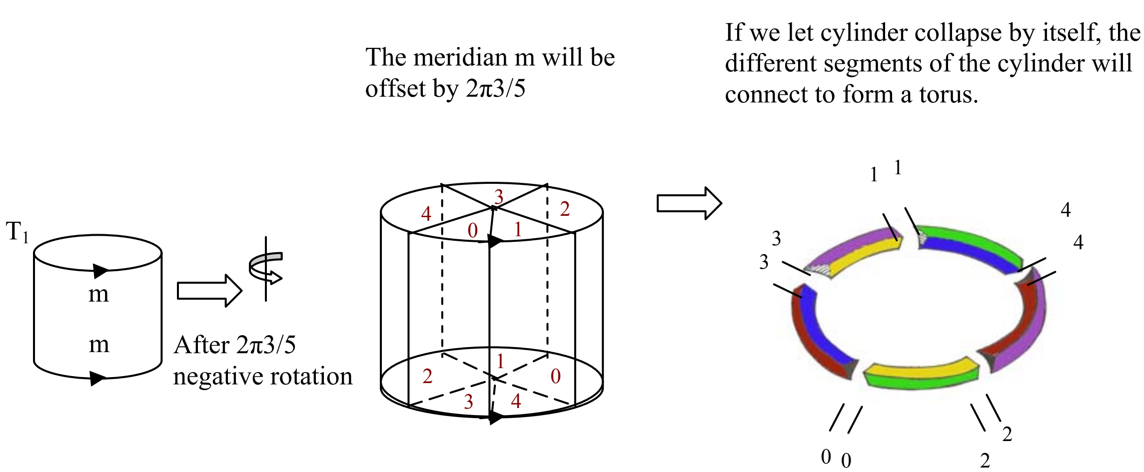
Let (p, q) = (5, 3)



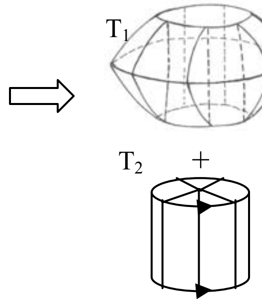
Cutting T_2 across a half – plane gives a solid cylinder with p disjoint sections of the knot (p, q) along its side.



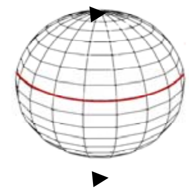
Cutting T_1 across a half – plane containing the meridian m gives a solid cylinder with identical ends and a copy of m around each end



Note that the same coloured portion was initially connected



Now if we slot T_2 (after $2\pi 3/5$ negative rotation) inside T_1 and rotate it $2\pi 3/5$ in the positive direction, the regions of upper cap and the lower cap will be identified. This is in fact the model of L(p, q) described in model 4.1. Thus, models 4.1 and 4.2 result in the same lens spaces without any discrepancy.



6. Link between definitions 4.1 and 4.3

From definition 4.3, Let S^3 be the set of complex numbers (z_0, z_1), where $z_0 = r_0 e^{i\theta_0}$, $z_1 = r_1 e^{i\theta_1}$ and $r_0^2 + r_1^2 = 1$ (fig 4.7). Further fixing $r_0 = (1 - r_1^2)^{1/2}$, (z_0, z_1) can be associated with three variables, $(r_1, \theta_0, \theta_1)$. We can combine the three variables to form a solid torus T. There are two special cases that we need to consider. When $r_1 = 0$, $r_0 = 1$, $(z_0, z_1) = (e^{i\theta_0}, 0)$ which is independent of θ_1 (T shrinks down to a latitude). When $r_1 = 1$, $r_0 = 0$, $(z_0, z_1) = (0, e^{i\theta_1})$ which is independent of θ_0 (Latitudes shrink down to points. T becomes a solid S^2. Fig 5.2).

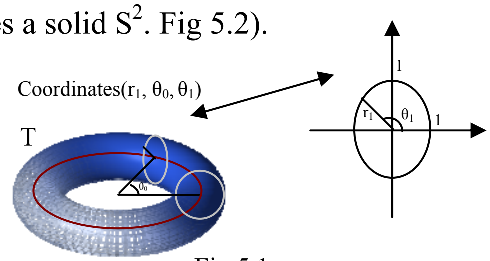


Fig 5.1

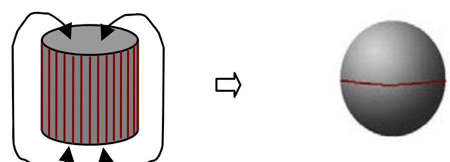


Fig 5.2

Consider S^3 under the $\mathbb{Z}_p$ action, $m\cdot(z_0, z_1) = (e^{2\pi i m/p}\cdot z_0, e^{2\pi i q m/p}\cdot z_1) = (r_0\cdot e^{i(\theta_0 + 2\pi m/p)}, r_1\cdot e^{i(\theta_1 + 2\pi m/p)})$. Therefore the $\mathbb{Z}_p$ action on T is $m\cdot(r, \theta_0, \theta_1) = (r, \theta_0+2\pi m/p, \theta_1+2\pi mq/p)$. The value of r is unaffected by the $\mathbb{Z}_p$ action. Lets restrict our attention on individual tori T_c, where $r = c$ $(0 \le c \le 1)$. $T = \bigcup_{0\le c\le 1} T_c$. When $0<c<1$, T_c can be represented as a square with identified edges, with the horizontal axis acting as the θ_1-axis and the vertical as the θ_0-axis, ranging from 0 to 2π (fig 5.3). The action of m on the point (θ_0, θ_1) results in the point $(\theta_0+2\pi m/p, \theta_1+2\pi mq/p)$. Thus $\mathbb{Z}_p$ actions are translations along lines with gradient 1/q. Consider the case of p = 5, q = 3 (fig 5.4). This will result in the fundamental domain (fig 5.5) which is a short tube with ends identified after a $2\pi q/p$ rotation (fig 5.6).

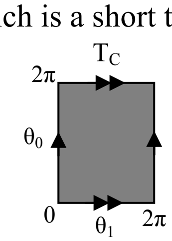


Fig 5.3

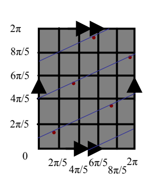


Fig 5.4

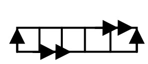


Fig 5.5

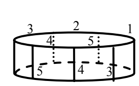


Fig 5.6

Let's reconsider the two special cases (with $\mathbb{Z}_p$ action): T_0 (the central circle) and T_1 (∂T with identification). T_0 is the circle when $r_1 = 0$, the action of m results in $\theta_0 \rightarrow \theta_0 + 2\pi m/p$, inducing a $2\pi m/p$ rotation. This will result in the fundamental domain (fig 5.7) and $T_0/\mathbb{Z}_p$ can be seen as an arc with identified endpoints. T_1 results when $r_1 = 1$ which is formed when the latitudes are identified to a point each. The action of m will result in $\theta_1 \rightarrow \theta_1 + 2\pi mq/p$ inducing a $2\pi mq/p$ rotation. If we partition it into p equal arcs, the identification will be as illustrated in fig 5.8.

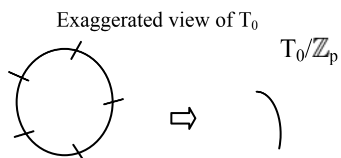


Fig 5.7

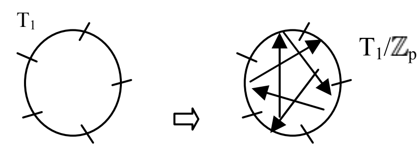


Fig 5.8

If we combine all the cases of T_c under $\mathbb{Z}_p$ action, it is evident that $S^3/\mathbb{Z}_p = \bigcup_{0\leq c\leq 1} T_c/\mathbb{Z}_p$. This is a collection of nested tubes with radii $0 \leq r < 1$ whose ends are identified after a $2\pi q/p$ rotation, and a boundary of latitude-segments which are identified to a single point. If we collapse the latitude-segment to a point, we get a lens form whose circular rim is divided into p segments, all identified together. The upper and lower caps of the lens form are identified after a $2\pi q/p$ rotation.

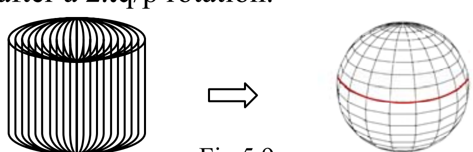


Fig 5.9

This is L(p, q) described in model 4.1. Thus, models 4.1 and 4.3 describe the same lens space. Combining the proof in section 5, we can conclude that the three models of lens spaces are consequently equivalent.

7. Acknowledgement

Special thanks to Associate Professor, Edward Teo from the National University of Singapore for his guidance.

8. References

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