# Weak Roman Domination

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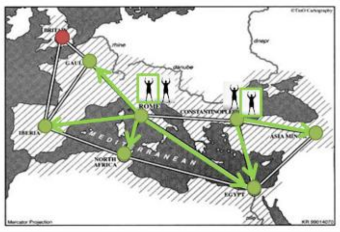
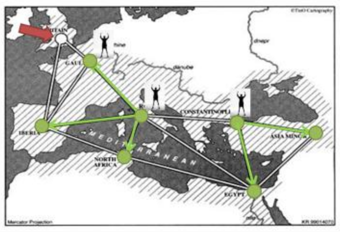
## Introduction

In the 4th century, Emperor Constantine of the Roman Empire was facing a deployment problem: He had to decide where to station his four field army legions to protect eight regions. This problem becomes known as the famous Roman Domination problem in graph theory.

In 2003, a new strategy called the Weak Roman Domination (WRD) is introduced. It helps Emperor Constantine to cut down the number of legions to be maintained while still defending the Roman Empire.



**Roman domination**



**Weak Roman domination**

(Modified from a graph adapted from the website of American Mathematics Association)

## Definitions

Let $G=(V,E)$ be a graph and $f$ be a function $f: V \rightarrow \{0,1,2\}$.

◆**Undefended Vertex**: A vertex $u$ with $f(u)=0$ is said to be undefended with respect to $f$ if it is not adjacent to a vertex with positive weight.

◆**Weak Roman Dominating Function**: The function $f$ is a Weak Roman Dominating Function (WRDF) if each vertex $u$ with $f(u)=0$ is adjacent to a vertex $v$ with $f(v)>0$ such that the function $f': V \rightarrow \{0,1,2\}$; defined by $f'(u)=1$, $f'(v)=f(v)-1$, $f'(w)=f(w)$, where $w \in V-\{u,v\}$ has no undefended vertex.

◆**Weight**: The weight of $f$ is $w(f) = \sum_{v\in V} f(v)$.

◆**Weak Roman Domination Number**: The Weak Roman Domination Number, denoted by $\gamma_r(G)$, is the minimum weight of a WRDF in $G$.

## Weak Roman Dominating Index

The introduction of **Weak Roman Dominating Index** helps us to simplify complex graphs when dealing with Weak Roman Domination problems. This index deals with the change in Weak Roman Domination Number when an edge is added or removed in a graph. It also corresponds to the real-life scenarios as people always want to find the most strategic positions to place their new transportation lines, roads, etc to gain the greatest economic interests.

Let $G$ be a graph and $x$, $y$ two non-adjacent vertices in $G$, The **Weak Roman Dominating Index** of $\{x, y\}$, denoted by $r(xy)$, is defined by $r(xy) = \gamma_r(G) - \gamma_r(G+xy)$.

***Proposition:*** *Let $G$ be a graph. For any pair of non-adjacent vertices in $G$, $0 \le r(xy) \le 1$.*

**Remark:** Both bounds are achievable. Following are the four cases in all.

**Case 1:** $f'(x)=0$, $f'(y)=0$
**Case 2:** $f'(x)>0$, $f'(y)>0$
} $r(xy)=0$

**Case 3:** $f'(x)=0$, $f'(y)=2$ or $f'(x)=2$, $f'(y)=0$
**Case 4:** $f'(x)=0$, $f'(y)=1$ or $f'(x)=1$, $f'(y)=0$
} $r(xy)=0$ or $1$

**Reference:**
1. **E.J. Cockayne, P.A. Dreyer, S.M. Hedetniemi, and S.T. Hedetniemi,** (2000) *Roman domination in graphs*, manuscript.
2. **Lim, Vanson.** (2007) *Roman and Weak Roman Domination in Graphs,* manuscript.
3. **Henninga, Michael A. and Hedetniemi, Stephen T.** (2003) *Defending the Roman Empire—A new strategy.* Discrete Mathematics 266, pp. 239-251.
4. **ReVelle, Charles S. and Rosing, Kenneth E.** (2000) *Defendens Imperium Romanum: A Classical Problem in Military Strategy*. The American Mathematical Monthly, Vol. 107, No. 7. Aug. - Sep., pp. 585-594.
5. **Stewart, Ian.** (1999) *Defend the Roman Empire*. Scientific American, December, pp. 136-138.

## Bounds of Weak Roman Domination Number

***Proposition***: *For any tree $T$ of order $n \ge 3$,* $2 \le \gamma_r(T) \le \left\lfloor \frac{2n}{3} \right\rfloor$

(This proposition is suggested by Vanson Lim in his 2007 NUS undergraduate paper, but without a proof)

**Proof Outline:** For the lower bound, a star of order $n$ will have a WRD number of 2. For the upper bound, mathematical induction on the diameter of the tree $D(T)$ will be used.

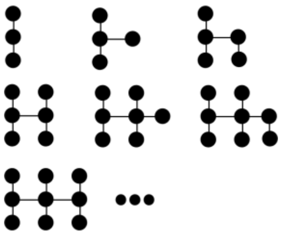
**Inductive hypothesis:** If $\gamma_r(T) \le \left\lfloor \frac{2n}{3} \right\rfloor$ for any tree $T$ with $k-3 \le D(T) \le k-1$

Then for any tree $T$ of $D(T)=k$, $\gamma_r(T) \le \left\lfloor \frac{2n}{3} \right\rfloor$

**Remark:**

◆This is the best upper bound we can get. For each order, this upper bound is achieved by a tree of the structure shown in the graph.

◆Given a tree $T$ of order $n \ge 3$, $\gamma_r(T) = \frac{2n}{3}$ if and only if $T$ has a structure in the first column

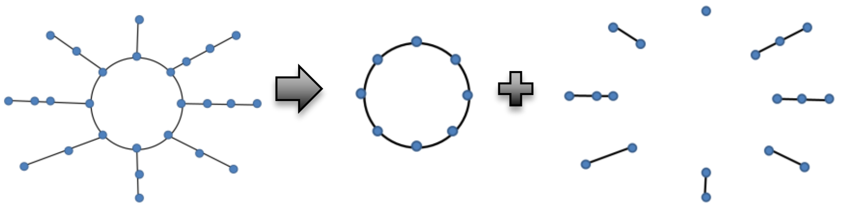


***Corollary*** *: For any connected graph $G$,* $1 \le \gamma_r(G) \le \left\lfloor \frac{2n}{3} \right\rfloor$

## Bounds of Weak Roman Domination Number for Unicyclic Graphs

***Proposition:*** *For $G \in \mathbb{C}_n$, $\gamma_r(G) \le \gamma_r(C_n)$ for $3 \le n \le 6$, and $\gamma_r(G) \le \left\lfloor \frac{2n-2}{3} \right\rfloor$, for $n \ge 7$.*

**Proof Outline:** The graph is split into two sub graphs, one of which is the cycle it contains.



$G$ $C_k$ $G/C_k$

$$\gamma_r(G) \le \gamma_r(C_k) + \gamma_r(G/C_k) = \left\lceil \frac{3k}{7} \right\rceil + \left\lfloor \frac{2(n-k)}{3} \right\rfloor \le \left\lfloor \frac{2n-2}{3} \right\rfloor \quad \text{for } k \le 4 \text{ or } k \ge 6$$

**Remark:** This upper bound is achieved when a unicyclic graph contains a cycle of $C_3, C_4, C_5, C_6$ or $C_8$

## Application

**Weak Roman Domination** strategy can be applied in:

◆Deciding the locations of ice-cream stands in a region

◆Placement of security guards in the museum

◆Setting up of branches in a foreign country for multinational companies

## Future Research

For future research, a comparison between Roman Domination and Weak Roman Domination strategy can be made. Some other generalized properties of weak Roman domination number and exact values of WDR number of certain types of graphs can also be explored.

**Photo Credit:** Unless otherwise stated, all graphs are self-drawn.