1 Introduction

1.1 Background

In the 4th century, Emperor Constantine of the Roman Empire was facing a deployment problem: He had to decide where to station his four field army legions to protect eight regions. This problem becomes known as the famous Roman Domination problem in graph theory. In 2003, a new strategy called the Weak Roman Domination (WRD) is introduced, which helps the Emperor Constantine to cut down the number of legions to be maintained while still defending the Roman Empire. In graph theoretic terminology, let $G=(V,E)$ be a graph and f be a function $f:V\rightarrow\{0,1,2\}$; a vertex u with $f(u)=0$ is said to be undefended with respect to f if it is not adjacent to a vertex with positive weight. The function f is a weak Roman dominating function (WRDF) if each vertex u with $f(u)=0$ is adjacent to a vertex v with $f(v)>0$ such that the function $f':V\rightarrow\{0,1,2\}$; defined by $f'(u)=1, f'(v)=f(v)-1, f'(w)=f(w)\ if\ w\in V-\{u,v\}$ has no undefined vertex. The weight of f is $w(f)$ = $\sum_{v\in V} f(v)$. The Weak Roman Domination number, denoted by $\gamma_r(G)$, is the minimum weight of a WRDF in G.

1.2 Objectives

1. To determine the Weak Roman Dominating index
2. To determine the bounds for $\gamma_r(G)$ in any general graph G
3. To determine the bounds for the Weak Roman Domination number $\gamma_r(G)$ in unicyclic graphs

1.3 Methodology

Though Weak Roman Domination (WRD) is proposed in 2003, not much research has been done on it. It fulfills the objective of modern industrial management concept of cutting down production cost and improved on efficiency. This project begins by focusing on Weak Roman Dominating index, which helps us to determine the importance of each edge in a graph; followed by an exploration on the most general property, namely the bounds for $\gamma_r(G)$.

2 Workings and Results **[All general definitions are included in the Appendix]**

2.1 Existing results

The following result was proved by Dreyer (2000) and Henning (2003) respectively:

Proposition 1: *For path P_n and cycle C_n of order n, $\gamma_R(P_n)=\gamma_R(C_n)=\left\lceil \frac{2n}{3} \right\rceil$.*

Proposition 2: *For path P_n and cycle C_n of order n, $\gamma_r(P_n)=\gamma_r(C_n)=\left\lceil \frac{3n}{7} \right\rceil$.*

The next observation follows readily from the definition.

Proposition 3: *If H is a spanning sub-graph of a graph G, then $\gamma_r(H) \geq \gamma_r(G)$.*

2.2 Weak Roman Dominating index

In practice, armies, utility operators etc are concerned about where to build new roads, pipelines etc so as to reduce the size of the army or reap the most economic benefits. As such, I introduced a new concept called the **Weak Roman Dominating index**. It is useful in simplifying specific Roman Domination problems. Let G be a graph and x, y two non-adjacent vertices in G, The **Weak Roman Dominating index** of $\{x, y\}$, denoted by $r(xy)$, is defined by $\gamma_r(xy) = \gamma_r(G) - \gamma_r(G+xy)$.

Proposition 4: *Let G be a graph. For any pair of non-adjacent vertices $\{x, y\}$ in G, $0 \leq r(xy) \leq 1$.*

Proof: We only need to prove that $r(xy) \leq 1$**.** Let $G'=G+xy$ and f' be a γ_r -function of G'. There are four cases to consider [Detailed proof is in the Appendix].

Case 1: $f'(x)=0$,$f'(y)=2$ or $f'(x)=2$,$f'(y)=0$. In this case that $r(xy)=0$ or 1.

Case 2: $f'(x) > 0$,$f'(y) > 0$. In this case $r(xy)=0$

Case 3: $f'(x)=0$,$f'(y)=0$. In this case $r(xy)=1$.

Case 4: $f'(x)=0$,$f'(y)=1$ or $f'(x)=1$,$f'(y)=0$. In this case $r(xy)=0$ or 1.

Summing up the above-mentioned cases of discussion on the upper bound, we have $0 \leq r(xy) \leq 1$

2.3 Bounds of Weak Roman Domination number

Firstly, we define the **diameter** of a graph G.

The **diameter** of a graph G, denoted by $D(G)$, is defined as

$$D(G) = \max\{d(u,v) \mid u, \text{v are in the } V\}.$$

Also, note that

$\lfloor a+b \rfloor \geq \lfloor a \rfloor + \lfloor b \rfloor$ for any real number a and b, $\left\lfloor \frac{2n}{3} \right\rfloor \geq \frac{2n-2}{3}$ for any positive integer n

These two conclusions are used in the following proof.

Proposition 5*: For any tree T of order $n \geq 3$,* $2 \leq \gamma_r(T) \leq \left\lfloor \frac{2n}{3} \right\rfloor$.

Proof: For the lower bound, no matter what the order n is, a star always has a weak Roman Domination number of 2. However, a tree of order $n \geq 3$ will never have a Weak Roman Domination number of 1. This is because if we choose a path of $v_0 e_0 v_1 e_1 v_2$ from the tree, since now we only have one army; the only possible case is that $f(v_1)=1, f(v_2)=f(v_0)=0$. The movement of the army from v_1 to either v_0 or v_2 will leave the other unprotected, which does not satisfy the definition of WRDF, hence $\gamma_r(T)\geq 2$

For the upper bound, mathematical induction is applied in the proof.

Case 1: $D(T)=2$. In this case the tree is actually a star, and it is obvious that

$$\gamma_r(T)=2=\left\lfloor \frac{2\times 3}{3}\right\rfloor \leq \left\lfloor \frac{2n}{3}\right\rfloor \text{ since } n\geq 3$$

Case 2: $D(T)=3$. In this case, a path $v_0e_0v_1e_1v_2e_2v_3$ of length 3 and without loss of generality is found. We can assume $d(v_2)\geq d(v_1)$.

If $d(v_2)\geq d(v_1)>2$, by removing edge e_1 we can get two isolated trees T_1 and T_2, both of which have a diameter of 2. By proposition, Hence

$$\gamma_r(T)\leq \gamma_r(T_1)+\gamma_r(T_2)=2+2=4=\left\lfloor \frac{2\times 6}{3}\right\rfloor \leq \left\lfloor \frac{2n}{3}\right\rfloor \text{ since } n\geq 6$$

Otherwise, $d(v_1)=2$, let $f(v_1)=1$ and $f(v_2)=2$, the graph now is a WRDF and $w(f)=1+2=3$. Hence we conclude that $\gamma_r(T)\leq w(f)=3\leq \left\lfloor \frac{2\times 5}{3}\right\rfloor \leq \left\lfloor \frac{2n}{3}\right\rfloor$ since $n\geq 5$

Case 3: $D(T)=4$. In this case, we can find a path $v_0e_0v_1e_1v_2e_2v_3e_3v_4$ of length 4 without the loss of generality, we can assume $d(v_3)\geq d(v_1)$.

If $d(v_3)>2$, we can remove v_3 together with all its neighboring end vertices as a whole, which forms a tree of diameter 2. Repeat this step until the tree decreases in diameter to some previously-discussed cases; or becomes a tree T' where $d(v_1)=d(v_3)=2$ and $d(v_2)\geq 2$. For the former case, it is already discussed previously. As for the latter case, we can construct a Weak Roman Dominating function f for every sub-case. We let n' be the order of T', and we let x and y be the number of branches with one vertex and two vertices respectively. Obviously, $x\geq 2, y\geq 0$. There are three cases to consider:

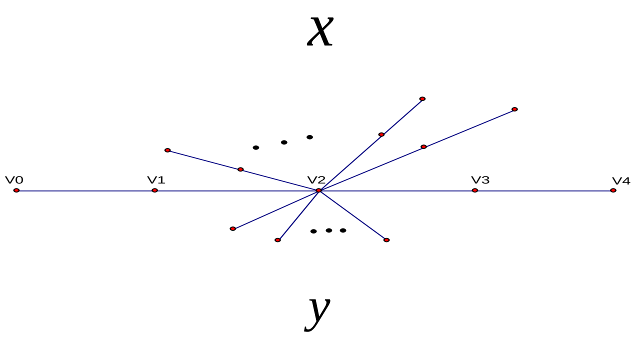

Figure 2.1

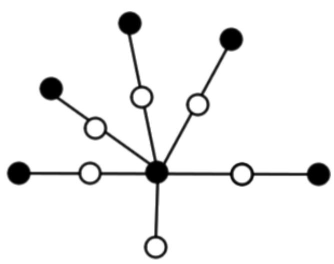
Figure 2.2 (Case ii)


Figure 2.3

i) $y \geq 2$. In this case, we let $f(v_2)=2$ and $f(v)=1$ for all end-vertices v of the branches with two vertices in T' (the neighbor vertices of v_2 are not considered as end-vertices). Thus we have $w(f)=2+x$. Noting that $2x+y+1=n'$, we have

$$\gamma_r(T') \leq w(f) = 2+x = 2+\frac{n'-y-1}{2} = \frac{n'-y+3}{2} \leq \frac{n'+1}{2} \leq \frac{2n'-2}{3} \leq \left\lfloor \frac{2n'}{3} \right\rfloor,$$

which is true since $n' = 2x+y+1 \geq 2\times 2+2+1=7$.

ii) $y = 1$. In this case, we let $f(v_2)=1$ and $f(v)=1$ for all end-vertices v of the branches with two vertices in T' (the neighbor vertices of v_2 are not considered as end-vertices). The graph now satisfies Weak Roman domination, and $w(f)=1+x$. By noting that $2x+2=n'$, we have

$$\gamma_r(T') \leq w(f) = 1+x = 1+\frac{n'-2}{2} = \frac{n'}{2} \leq \frac{2n'-2}{3} \leq \left\lfloor \frac{2n'}{3} \right\rfloor,$$

which is true since $n' = 2x+2 \geq 2\times 2 = 4$.

iii) $y = 0$. In this case, we let $f(v_2)=1$ and $f(v)=1$ for all end-vertices v of the branches with two vertices in T'. The graph now satisfies weak Roman domination, and $w(f)=1+x$. By noting that $2x+1=n'$, we have

$$\gamma_r(T') \leq w(f) = 1+x = 1+\frac{n'-1}{2} = \frac{n'+1}{2} \leq \left\lfloor \frac{2n'}{3} \right\rfloor,$$

which is true since $n' = 2x+1 \geq 2\times 2+1 = 5$.

Hence we conclude that $\gamma_r(T) \leq \left\lfloor \frac{2n}{3} \right\rfloor$ when $D(T)=4$

Inductive hypothesis: If $\gamma_r(T) \leq \left\lfloor \frac{2n}{3} \right\rfloor$ for any tree T where $k-3 \leq D(T) \leq k-1$, then for any tree T of $D(T)=k$, $\gamma_r(T) \leq \left\lfloor \frac{2n}{3} \right\rfloor$. To show this: For a given tree T of $D(T)=k$, we first find one of its longest path $v_0e_0v_1e_1v_2e_2\cdots v_{k-3}e_{k-3}v_{k-2}e_{k-2}v_{k-1}e_{k-1}v_k$. Then we remove edge e_{k-3}. Since there is only one path linking a vertex to another in any tree, removing an edge indicates that these two vertices are no longer linked by edges and other vertices. This results in two disjoint trees T_1 and T_{a_1}:

T_{a_1} contains the path $v_{k-2}e_{k-2}v_{k-1}e_{k-1}v_k$. $d(v_k, v_{k-2})=2$ implies that $D(T_{a_1}) \geq 2$. Since we have chosen the longest path in T, we have $D\left(T_{a_1}\right)\le 4$. Hence $2\le D\left(T_{a_1}\right)\le 4$ and T_{a_1} falls in the base cases mentioned above.

Let $v\left(T_{a_1}\right)$ denote the order of T_{a_1}. We have $\gamma_r\left(T_{a_1}\right)\le\left\lfloor\frac{2v\left(T_{a_1}\right)}{3}\right\rfloor$.

T_1 contains path $v_0e_0v_1e_1v_2e_2\cdots e_{k-2}v_{k-3}$. $d\left(v_0v_{k-3}\right)=k-3$ implies that $D(T_1)\ge k-3$. Also, $D(T_1)\le D(T)=k$. Thus $k-3\le D(T_1)\le k$. If $D(T_1)=k$, we notice that there are fewer paths of length k in T_1 than in T since at least path $v_0e_0v_1e_1v_2e_2\cdots v_{k-3}e_{k-3}v_{k-2}e_{k-2}v_{k-1}e_{k-1}v_k$ no longer exists in T_1. Hence if $D(T_1)=k$, repeat steps 1 and 2 such that at the i^{th} repetition of steps 1 and 2, we divide T_{i-1} into T_i and T_{a_i}. As the number of path of length k is finite and this number decreases each time when we apply step 1 and 2; after s repeats, $D(T_s)$ will for the first time be smaller than k, and we have $k-3\le D(T_s)\le k-1$.

By the inductive hypothesis, we have $\gamma_r\left(T_s\right)\le\left\lfloor\frac{2v\left(T_s\right)}{3}\right\rfloor$. Thus, for graph T of $D(T)=k$, we have

$$\gamma_r(T)\le\gamma_r(T_s+\sum_1^s T_{a_i})=\gamma_r\left(T_s\right)+\sum_{i=1}^s\gamma_r\left(T_{a_i}\right)$$
$$\le\left\lfloor\frac{2v\left(T_s\right)}{3}\right\rfloor+\sum_{i=1}^s\left\lfloor\frac{2v\left(T_{a_i}\right)}{3}\right\rfloor\le\left\lfloor\frac{2v\left(T_s\right)+2\sum_1^s v\left(T_{a_i}\right)}{3}\right\rfloor=\left\lfloor\frac{2v(T)}{3}\right\rfloor$$

Remark 1: This upper bound can be achieved by constructing trees of following structures.

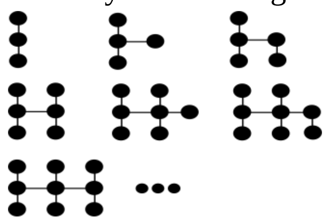

Figure 2.4

Remark 2: Given a tree T of order $n\ge 3$, $\gamma_r(T)=\frac{2n}{3}$ if and only if T has a structure like the left most ones shown in Figure 2.4

Proof: Sufficiency is shown directly by Proposition 6. For necessity, given $2\le D(T)\le 4$, only $\gamma_r(P_3)=\frac{2n}{3}$ .Only when $T_{a_i}=P_3$ for all $1\le i\le m$ will we have $\gamma_r(T)=\frac{2n}{3}$.

Corollary 6*: For any connected graph* G, $1\le\gamma_r\left(G\right)\le\left\lfloor\frac{2n}{3}\right\rfloor$.

Proof: By **Proposition 3,** we know that $\gamma_r\left(G\right)\le\gamma_r\left(H\right)$, and from **Proposition 5** we have $\gamma_r\left(H\right)\le\left\lfloor\frac{2n}{3}\right\rfloor$.

Hence $1 \le \gamma_r(G) \le \left\lfloor \frac{2n}{3} \right\rfloor$ □

2.4 Bounds for $\gamma_r(G)$ in unicyclic graphs

Proposition 7: *For* $G \in \mathbb{C}_n$, $\gamma_r(G) \le \gamma_r(C_n)$ *for* $3 \le n \le 6$, *and* $\gamma_r(G) \le \left\lfloor \frac{2n-2}{3} \right\rfloor$, *for* $n \ge 7$.

Proof: It is obvious that the conclusion holds for $3 \le n \le 6$.

For unicyclic graph $G(V,E)$ such that $G \in \mathbb{C}(n,k)$, with $n \ge 7$ and a cycle of order k, we name the vertices on the cycle $v_1, v_2, ..., v_k$. The open neighborhood of v_i $(i = 1,2,...,k)$ is $N(v_i) = \{u \in V \mid uv_i \in E\}$. Let $S_i = \{v_i u \mid u \in N(v_i), u \ne v_{i+1}, u \ne v_{i-1}\}$, $i = 1,2,...,k$, and $v_{k+1} = v_1$. Consider a graph $G'(V,E')$, where $E' = E - \bigcup_{i=1}^{k} S_i$. Hence G' is a spanning graph of G, due to Proposition 4, $\gamma_r(G) \le \gamma_r(G')$. Let C_k be the cycle formed by $v_1, v_2, ...v_k$, thus $\gamma_r(G') = \gamma_r(C_k) + \gamma_r(G'/C_k)$. Note that G'/C_k is a forest. Let $G'/C_k = \bigcup T_i$, where T_i are all trees. Hence,

$$\gamma_r(G'/C_k) = \sum \gamma_r(T_i) = \sum \left\lfloor \frac{2v(T_i)}{3} \right\rfloor \le \left\lfloor \sum \frac{2v(T_i)}{3} \right\rfloor = \left\lfloor \frac{2(n-k)}{3} \right\rfloor.$$

Note that $\gamma_r(C_k) = \left\lceil \frac{3k}{7} \right\rceil$, when $k \ge 6$ or $k \le 4$, $\gamma_r(C_k) = \left\lceil \frac{3k}{7} \right\rceil \le \left\lfloor \frac{2(k-1)}{3} \right\rfloor$. Hence

$$\gamma_r(G) \le \gamma_r(G') = \gamma_r(C_k) + \gamma_r(G'/C_k) \le \left\lfloor \frac{2(k-1)}{3} \right\rfloor + \left\lfloor \frac{2(n-k)}{3} \right\rfloor \le \left\lfloor \frac{2(n-1)}{3} \right\rfloor$$

Now we will prove when $k = 5$, $\gamma_r(G) \le \left\lfloor \frac{2n-2}{3} \right\rfloor$ still holds.

When $k = 5, \gamma_r(C_k) = \left\lceil \frac{3k}{7} \right\rceil = 3$, while $\left\lfloor \frac{2(k-1)}{3} \right\rfloor = 2$,

$$\gamma_r(G) \le \gamma_r(G') = \gamma_r(C_k) + \gamma_r(G'/C_k) = 3 + \left\lfloor \frac{2(n-5)}{3} \right\rfloor = \left\lfloor \frac{2n-1}{3} \right\rfloor.$$

Note that $\left\lfloor \frac{2n-1}{3} \right\rfloor = \left\lfloor \frac{2n-2}{3} \right\rfloor$ when $n \equiv 0,1 \pmod 3$. When $n \equiv 2 \pmod 3$, suppose f is the WRDF of G such that $\gamma_r(G) = \left\lfloor \frac{2n-1}{3} \right\rfloor$. G must follow the pattern in Figure 3.6 due to Proposition 6.

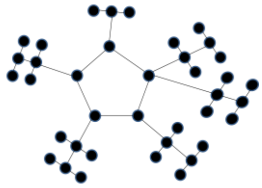

Figure 2.5

Since $n \ge 7$, hence $\bigcup_{i=1}^{5} S_i \ne \varnothing$. Due to proposition 6, $\exists v_i \in \{v_1,\ v_2,\ v_3,\ v_4,\ v_5\}$ such that $f(u)=2,$ where $u \in S_i$. Hence there exists a WRDF f' of G, with

$$f'(v)=\begin{cases} 0, & v=v_i,\ v_{i+1} \text{ and } v_{i-2}; \\ 1, & v=v_{i+1} \text{ and } v_{i-1}; \\ f(v), & v \notin \{v_i, i=1,2,\dots,5\} \end{cases}$$

$w(f')=2+\left\lfloor \frac{2(n-5)}{3} \right\rfloor \le \left\lfloor \frac{2(n-5)}{3}+2 \right\rfloor = \left\lfloor \frac{2n-4}{3} \right\rfloor < \left\lfloor \frac{2n-1}{3} \right\rfloor$. That's a contradiction with $\gamma_r(G)=\left\lfloor \frac{2n-1}{3} \right\rfloor$.

Hence $\gamma_r(G)<\left\lfloor \frac{2n-1}{3} \right\rfloor$. i.e. $\gamma_r(G)\le\left\lfloor \frac{2n-2}{3} \right\rfloor$. Thus the proof is completed.

Remark: This upper bound is achieved when a unicyclic graph contains a cycle of C_3, C_4, C_5, C_6 or C_8

3 Conclusions

Through the project, several propositions have been proved regarding WRD, which are fundamental results of this strategy. They will help us to apply this strategy later in real-life situations. Upper bounds of WRD are suggested to avoid waste of resources. Also, unicyclic graphs are good models of many real-life scenarios, especially for a company dealing with setting up branches in both core and peripheral areas. Furthermore, WRD can help people to design pipelines, build roads etc.

For further research, on one hand, I will try to find some other generalized properties of weak Roman domination number and find the exact values of weak Roman domination number of certain types of graphs. On the other hand, I will focus on some concrete example where weak Roman domination strategy can be applied in real-life situations and show how can it help people in decision-making.

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Appendices

1. Definitions and notations

1.1 General definitions and notations

A **graph** G consists of a non-empty finite set $V(G)$ of vertices together with a finite set $E(G)$ (possibly empty) of edges such that:

1. each edge joins two *distinct* vertices in $V(G)$ and

2. any two distinct vertices in $V(G)$ are joined by *at most one* edge.

The number of vertices in G, denoted by (G), is called the **order** of G.

Let u, v be any two vertices in G. They are said to be **adjacent** if they are joined by an edge, say, e in G. We also write $e=uv$ or $e=vu$ (the ordering of u and v in the expression is immaterial), and we say that

1. u is a **neighbor** of v and vice versa,

2. the edge e is **incident with** the vertex u (and v) and

3. u and v are the two **ends** of e.

The set of all neighbors of v in G is denoted by (v); that is,

$$N(v) = \{x | x \text{ is a neighbor of } v\}.$$

The **degree** of v in G, denoted by $d\ (v)$, is defined as the number of edges incident with v. The vertex v is called an **end-vertex** if $d\ (v)\ =1$.

A **path** in a graph G is an alternating sequence of vertices and edges beginning and ending at vertices:

$$v_0 e_0 v_1 e_1 v_2 \ldots v_{k-1} e_{k-1} v_k$$

where $k \geq 1$, e_i is incident with v_i and v_{i+1}, for each $i=0,1,\ldots,k-1$, and the vertices v_i's and edges e_i's need to be distinct. The **length** of the path above is defined as k, which is the number of occurrences of edges in the sequence.

A graph G is said to be **connected** if every two vertices in G are joined by a path, and **disconnected** if it is not connected.

The **distance** from u to v, denoted by $d\ (u,\ v)$, is defined as the *smallest length* of all u−v paths in G. (Note that $d\ (v)$ denotes the degree of v in G.)

Let P_n denote a **path** of n vertices, $P_n = v_1v_2\ldots v_n$, and C_n a **cycle** of n vertices, $C_n=v_1v_2\ldots v_nv_1$.

Notice that we have two definitions for path. What a 'path' really means should be clear from the context when it is mentioned.

A graph H is called a **subgraph** of graph G if $V(H) \subseteq V(G)$ and $E\ (H) \subseteq E(G)$. A subgraph H of a graph G is said to be **spanning** if $V(H) = V(G)$.

A **bipartite graph** is a graph in which vertices are decomposed into two sets such that no two graph vertices within the same set are adjacent.

1.2 Roman Domination

Let $G=(V,E)$ be a graph. A **Roman Dominating function** is a function $f\colon V \rightarrow \{0,1,2\}$ such that every vertex v for which $f(v) = 0$ has a neighbor u with $f(u) = 2$.

The **weight** of a Roman dominating function f is $w(f) = \sum_{v\in V} f(v)$. This corresponds to the total number of army units required under a specific deployment scheme.

We are interested in finding Roman dominating function(s) of minimum weight for a particular graph. It makes sense in the army placement context, because we want to minimize the number of army units needed to secure a particular set of given regions.

A Roman dominating function of minimum weight among all the possible Roman dominating functions is called a **γ_R-function**. The **Roman Domination number** of a graph G, denoted by $\gamma_R(G)$, is the weight of a γ_R-function – the minimum weight of all possible Roman dominating functions.

1.3 Weak Roman Domination

The function f is a **weak Roman dominating function** (WRDF) if each vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) > 0$ such that the function f': $V \rightarrow \{0,1,2\}$, defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u,v\}$, has no undefended vertex. The **weight** of f is $w(f) = \sum_{v \in V} f(v)$.

The **Weak Roman domination number**, denoted by $\gamma_r(G)$, is the minimum weight of a WRDF in G.

2. Detailed proof to Proposition 4

Proof: We need only to prove that $r(xy) \leq 1$. Let G'=G+xy and f' be a γ_r -function of G'. There are four cases to consider.

Case 1: $f'(x)$=0 ,$f'(y)$ =2 or $f'(x)$=2 ,$f'(y)$ =0

Without loss of generality, assume that $f'(x)$ =0 and $f'(y)$ =2, and define f: $V \rightarrow \{0,1,2\}$:

$$f(v) = \begin{cases} f'(v), & v \neq x, \\ 1, & v = x \end{cases}$$

Then f is a Weak Roman dominating function of G as removing edge xy only raises the possibility that vertex x may be unprotected, if f' for G' is to be used for G. Under the definition for the Weak Roman dominating function, for any unprotected vertex, it is adjacent to at least one vertex where there is at least one army and the movement of this army to the unprotected vertex will leave no unprotected vertex. Since now $f'(x)$=0, it indicates that an army from either y or other adjacent vertex to x should move to x. If that army comes from a vertex other than y, removing edge xy will not affect the Weak Roman dominating number of this graph. When that army comes from y and now we remove the edge xy, simply adding one more army to this vertex x will resolve the issue – in this way, all vertices are again protected, with an increase of one in Weak Roman domination number.

Clearly, $\gamma_r(f)= \gamma_r(G')+1$. Thus $\gamma_r(G) \leq \gamma_r(f)= \gamma_r(G')+1$. It follows that

$r(xy) = \gamma_r(G) - \gamma_r(G') \leq 1$.

Case 2: $f'(x) > 0$,$f'(y) > 0$

In this case we show that edge xy does not affect $\gamma_r(G)$. This is because for any movement involving armies from either x or y (let's say x), the movement of the army to a adjacent vertex with zero army (if there exists such vertices) will not leave x unprotected, regardless of whether edge xy exists or not. Hence edge xy does not affect $\gamma_r(G)$ and hence $r(xy)$=0

Case 3: $f'(x)$=0 ,$f'(y)$=0

In this case both x and y have to be adjacent to at least one vertex with armies. There are three sub cases.

i) If x and y are adjacent to two different vertices with positive weight, then removing edge xy will not affect $\gamma_r(G)$. The reasons are as follows: Let x be adjacent to u with $f'(u) > 0$ and y be adjacent to v with $f'(v) > 0$ and f' be a γ_r–function of G'. When edge xy is removed, a movement of an army from u to x will still make all the vertices protected, since now y is still adjacent to a vertex with positive weight, namely v, and vice versa. Hence $r(xy)$=0.

ii) If x and y are adjacent to the same vertex with positive weight of 2, then $r(xy)$=0. This is true since for any movement of army, the common neighbor of x and y will at least has one army left. Hence removing edge xy will still make x and y remain protected by their common neighbor.

iii) If x and y are adjacent to the same vertex with positive weight of 1, then $r(xy)$ =1. When x and y are only adjacent to the same vertex, as shown in Figure 1

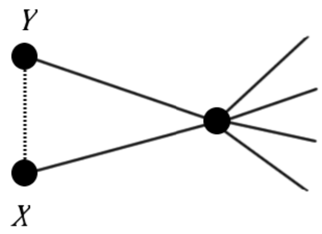

Figure 1

If G’ is a weak Roman domination graph, it indicates that when the only army moves from the common neighbor of x and y to either x or y, all the vertices are protected. When edge xy is removed, simply adding one more army to either x, y or their common neighbor will resolve the issue – in this way, all vertices are again protected, with an increase of one in weak Roman domination number, hence in this case $r(xy)$ =1.

Case 4: $f'(x)$=0 ,$f'(y)$ =1 or $f'(x)$=1 ,$f'(y)$ =0

Case 4 is similar to case 1. Without loss of generality, we let $f'(x)$=0 ,$f'(y)$ =1, and define $f: V \rightarrow \{0,1,2\}$:

$$f(v) = \begin{cases} f'(v), & v \neq x, \\ 1, & v = x \end{cases}$$

By similar argument as in case 1, Clearly, $\gamma_r(f) = \gamma_r(G')+1$. Thus $\gamma_r(G) \leq \gamma_r(f) = \gamma_r(G')+1$. It follows that $r(xy) = \gamma_r(G) - \gamma_r(G') \leq 1$.

Summing up the aforementioned several cases of discussion on the upper bound, we have $0 \leq r(xy) \leq 1$

□